

# Optimal Tableau Method for Constructive Satisfiability Testing and Model Synthesis in the Alternating-time Temporal Logic $\text{ATL}^+$

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## Abstract

We develop a sound, complete and practically implementable tableau-based decision method for constructive satisfiability testing and model synthesis for the fragment  $\text{ATL}^+$  of the full Alternating time temporal logic  $\text{ATL}^*$ . The method extends in an essential way a previously developed tableau-based decision method for  $\text{ATL}$  and works in  $2\text{EXPTIME}$ , which is the optimal worst-case complexity of the satisfiability problem for  $\text{ATL}^+$ . We also discuss how suitable parameterizations and syntactic restrictions on the class of input  $\text{ATL}^+$  formulae can reduce the complexity of the satisfiability problem.

keywords: alternating-time temporal logics,  $\text{ATL}^+$ , decision procedure, model synthesis, satisfiability, tableaux

## 1 Introduction

The Alternating-time temporal logic  $\text{ATL}^*$  was introduced and studied in [1] as a multi-agent extension of the branching time temporal logic  $\text{CTL}^*$ , where the path quantifiers are generalized to “strategic quantifiers”, indexed with coalitions of agents  $A$  and ranging existentially over collective strategies of  $A$  and then universally over all paths (computations) enabled by the selected collective strategy of  $A$ .  $\text{ATL}^*$  was proposed as logical framework for specification and verification of properties of open systems modelled as concurrent game models, in which all agents effect state transitions collectively, by taking simultaneous actions at each state. The language of  $\text{ATL}^*$  allows expressing statements of the type “Coalition  $A$  has a collective strategy to guarantee the satisfaction of

the objective  $\Phi$  on every play enabled by that strategy". The syntactic fragment ATL of  $\text{ATL}^*$  allows only state formulae, where all occurrences of temporal operators must be immediately preceded by strategic quantifiers. The fragment  $\text{ATL}^+$  of  $\text{ATL}^*$  extends ATL by allowing any Boolean combinations of ATL objectives in the scope of a strategic quantifier. It is considerably more expressive than ATL, which is reflected in the high – 2EXPTIME – worst-case complexity lower bound of the satisfiability problem for  $\text{ATL}^+$  (inherited from the lower bound for  $\text{CTL}^+$ , see [10]) as opposed to the EXPTIME-completeness of the satisfiability problem for ATL [5, 13]. The matching 2EXPTIME upper bound is provided by the automata-based method for deciding satisfiability in the full  $\text{ATL}^*$ , developed in [12].

The contribution of this paper is the development of a sound, complete and terminating tableau-based decision method for constructive satisfiability testing of  $\text{ATL}^+$  formulae. We also claim that our approach is intuitive and conceptually simple, as well as practically implementable and even manually usable, despite the inherently high worst-case complexity of the problem. The tableau method presented here is based on the general methodology going back to [11] (for PDL), [14] (for LTL) and [2, 6] (for CTL), further adapted for ATL in [9] to which the reader is referred for more details. A recent implementation of such a method is reported in [4]. The tableau method for  $\text{ATL}^+$  is an essential extension of the one for ATL, as it has to deal with much more complex (and computationally expensive) path objectives that can be assigned to the agents. It is also rather different from the above mentioned automata-based method in [12].

The paper is structured as follows. In Section 2 we offer brief technical preliminaries on concurrent game models, syntax and semantics of  $\text{ATL}^*$  and  $\text{ATL}^+$ . Section 3 develops the technical machinery needed for the presentation of the tableau method itself in Section 4. In Section 5 we prove the soundness of the tableau method, whereas in Section 6 we prove its completeness and demonstrate with examples how satisfying models can be extracted from the final open tableau. We also estimate the worst-case complexity of the procedure. In Section 7 we offer a brief comparison with the automata-based method in [12].

## 2 Preliminaries

We assume that the reader has basic familiarity with the branching time logic  $\text{CTL}^*$ , see e.g. [7]. Also, basic knowledge on  $\text{ATL}^*$  [1] and the tableaux-based decision procedure for ATL in [9], on which this paper builds, would be beneficial.

### 2.1 Concurrent game models, strategies and co-strategies

For technical reasons that will become clearer later in the soundness and the completeness proofs, we define a more general, *non-deterministic* version of the concurrent game structure with respect to [1]. For the moment, we can say that the basic idea is avoiding several definitions of the notion of Realization Witness

Trees for very similar structures (models, tableaux and Hintikka structures). Note that the very notion of tableau will be defined as a non-deterministic labelled CGS (see the beginning of Section 4).

Notation: given a set  $X$ , we denote the power set of  $X$  by  $\mathcal{P}(X)$ .

**Definition 2.1.** A *(non-deterministic) concurrent game structure* (CGS) is a tuple

$$\mathcal{S} = (\mathbb{A}, \text{St}, \{\text{Act}_a\}_{a \in \mathbb{A}}, \{\text{act}_a\}_{a \in \mathbb{A}}, \text{out})$$

comprising:

- a finite, non-empty set of *players (agents)*  $\mathbb{A} = \{1, \dots, k\}$
- a non-empty set of *states*  $\text{St}$ ,
- a set of actions  $\text{Act}_a \neq \emptyset$  for each  $a \in \mathbb{A}$ .

For any  $A \subseteq \mathbb{A}$  we denote  $\text{Act}_A := \prod_{a \in A} \text{Act}_a$  and use  $\sigma_A$  to denote a tuple from  $\text{Act}_A$ . In particular,  $\text{Act}_{\mathbb{A}}$  is the set of all possible *action profiles* in  $\mathcal{S}$ .

- for each  $a \in \mathbb{A}$ , a map  $\text{act}_a : \text{St} \rightarrow \mathcal{P}(\text{Act}_a) \setminus \{\emptyset\}$  defining for each state  $s$  the actions available to  $a$  at  $s$ ,
- a *transition relation*  $\text{out} \subseteq \text{St} \times \text{Act}_{\mathbb{A}} \times \text{St}$ .

Whenever  $\langle s, \sigma_{\mathbb{A}}, s' \rangle \in \text{out}$ , for  $\sigma_{\mathbb{A}} = \langle \sigma_1, \dots, \sigma_k \rangle$ , then  $\sigma_a \in \text{act}_a(s)$  for every  $a \in \mathbb{A}$ . Given a pair  $\langle s, \sigma_{\mathbb{A}} \rangle$ , the set of states  $s' \in \text{St}$  such that  $\langle s, \sigma_{\mathbb{A}}, s' \rangle \in \text{out}$  is denoted  $\text{out}(s, \sigma_{\mathbb{A}})$  and called the set of *successor (outcome) states of  $\sigma_{\mathbb{A}}$  at  $s$* .

When  $\text{out}(s, \sigma_{\mathbb{A}})$  is a singleton, the CGS is said to be *deterministic*. In such cases, by a slight abuse of notation we will use  $\text{out}(s, \sigma_{\mathbb{A}})$  to denote a state  $s'$  rather than the singleton  $\{s'\}$ .

**Definition 2.2.** 1. Given a set of formulae (of some language)  $\Theta$ , a CGS  $\mathcal{S}$  with a state space  $\text{St}$  is *state-labelled by  $\Theta$*  if there is a mapping  $l : \text{St} \rightarrow \mathcal{P}(\Theta)$  assigning to every state in  $\mathcal{S}$  a set of formulae from  $\Theta$ , called the *label* of that state.

2. A *concurrent game model* (CGM) is a *deterministic* CGS state-labelled by a fixed set of atomic propositions  $\text{Prop}$ , i.e., a tuple

$$\mathcal{M} = (\mathbb{A}, \text{St}, \{\text{Act}_a\}_{a \in \mathbb{A}}, \{\text{act}_a\}_{a \in \mathbb{A}}, \text{out}, \text{Prop}, L) \text{ where}$$

- $(\mathbb{A}, \text{St}, \{\text{Act}_a\}_{a \in \mathbb{A}}, \{\text{act}_a\}_{a \in \mathbb{A}}, \text{out})$  is a deterministic CGS,
- $\text{Prop}$  is a set of *atomic propositions*, and
- $L : \text{St} \rightarrow \mathcal{P}(\text{Prop})$  is a *(state-)labelling function*.

Concurrent game models represent multi-agent discrete transition systems that function as follows. At any moment the system is in a given state, where each agent selects an action from those available to him at that state. All agents execute their actions synchronously and the combination of these actions

together with the current state determines a transition to a unique successor state in the model. A *play* in a CGM is an infinite sequence of subsequent successor states, i.e., an infinite sequence  $s_0 s_1 \dots \in \text{St}^\omega$  of states such that for each  $i \geq 0$  there exists an action profile  $\sigma_{\mathbb{A}} = \langle \sigma_1, \dots, \sigma_k \rangle$  such that  $\text{out}(s_i, \sigma_{\mathbb{A}}) = s_{i+1}$ . A *history* is a finite prefix of a play. We denote by  $\text{Plays}_{\mathcal{M}}$  and  $\text{Hist}_{\mathcal{M}}$  respectively the set of plays and set of histories in a CGM  $\mathcal{M}$ . For a state  $s \in \text{St}$  we define  $\text{Plays}_{\mathcal{M}}(s)$  and  $\text{Hist}_{\mathcal{M}}(s)$  as the set of plays and set of histories with initial state  $s$ . Given a sequence of states  $\lambda$ , we denote by  $\lambda_0$  its initial state, by  $\lambda_i$  its  $(i+1)$ th state, by  $\lambda_{\leq i}$  the prefix  $\lambda_0 \dots \lambda_i$  of  $\lambda$  and by  $\lambda_{\geq i}$  the suffix  $\lambda_i \lambda_{i+1} \dots$  of  $\lambda$ . When  $\lambda = \lambda_0 \dots \lambda_\ell$  is finite, we say that it has length  $\ell$  and write  $|\lambda| = \ell$ . Further, we put  $\text{last}(\lambda) = \lambda_\ell$ .

For any coalition  $A \subseteq \mathbb{A}$ , a given CGM  $\mathcal{M}$  and state  $s \in \text{St}$ , an *A-co-action* at  $s$  in  $\mathcal{M}$  is a mapping  $\text{Act}_A^c : \text{Act}_A \rightarrow \text{Act}_{\mathbb{A} \setminus A}$  that assigns to every collective action of  $A$  at the state  $s$  a collective action at  $s$  for the complementary coalition  $\mathbb{A} \setminus A$ .

We use  $\text{act}_A(s)$  to denote the set of all  $A$ -actions that can be played by the coalition  $A$  at state  $s$ , i.e.  $\text{act}_A(s) = \Pi_{a \in A} \text{act}_a(s)$ . We also use  $\text{act}_A^c(s)$  to denote the set of all  $A$ -co-actions available at state  $s$  and  $\sigma_A^c$  for an element of this set.

A (*perfect recall*) *strategy* for an agent  $a$  in  $\mathcal{M}$  is a mapping  $F_a : \text{Hist}_{\mathcal{M}} \rightarrow \text{Act}_a$  such that for all  $h \in \text{Hist}_{\mathcal{M}}$  we have  $F_a(h) \in \text{act}_a(\text{last}(h))$ . Intuitively, it assigns an admissible action for agent  $a$  after any history  $h$  of the game. We denote by  $\text{Strat}_{\mathcal{M}}(a)$  the set of all strategies of agent  $a$ . A (collective) strategy of a set (*coalition*) of agents  $A \subseteq \mathbb{A}$  is a tuple  $(F_a)_{a \in A}$  of strategies, one for each agent in  $A$ . When  $A = \mathbb{A}$  this is called a *strategy profile*. We denote by  $\text{Strat}_{\mathcal{M}}(A)$  the set of collective strategies of coalition  $A$ . A play  $\lambda \in \text{Plays}_{\mathcal{M}}$  is *consistent with a collective strategy*  $F_A \in \text{Strat}_{\mathcal{M}}(A)$  if for every  $i \geq 0$  there exists an action profile  $\sigma_{\mathbb{A}} = \langle \sigma_1, \dots, \sigma_k \rangle$  such that  $\text{out}(\lambda_i, \sigma_{\mathbb{A}}) = \lambda_{i+1}$  and  $\sigma_a = F_a(\lambda_{\leq i})$  for all  $a \in A$ . The set of plays with initial state  $s$  that are consistent with  $F_A$  is denoted  $\text{Plays}_{\mathcal{M}}(s, F_A)$ .

Likewise, a (*perfect-recall*) *A-co-strategy* in  $\mathcal{M}$  for a coalition of agents  $A$  (possibly reduced to just one agent  $a$ ) is a mapping  $F_{\mathbb{A} \setminus A} : \text{Hist}_{\mathcal{M}} \times \text{Strat}_{\mathcal{M}}(A) \rightarrow \text{Act}_{\mathbb{A} \setminus A}$  that assigns to each  $h \in \text{Hist}_{\mathcal{M}}$  and every collective strategy  $F_A \in \text{Strat}_{\mathcal{M}}(A)$  an  $A$ -co-action  $F_{\mathbb{A} \setminus A}(h, F_A) \in \text{act}_{\mathbb{A} \setminus A}^c(\text{last}(h))$ .

## 2.2 The logic ATL\* and fragments

The logic ATL\* is a multi-agent extension of CTL\* with *strategic quantifiers*  $\langle\langle A \rangle\rangle$  indexed with coalitions  $A$  of agents. There are two types of formulae in ATL\*: *state formulae*, that are evaluated at states, and *path formulae*, that are evaluated on plays. To simplify the presentation we will work with formulae in negation normal form over a fixed set **Prop** of atomic propositions and primitive temporal operators *Always*  $\Box$  and *Until*  $\mathcal{U}$ . The syntax of the full language ATL\* and its fragments  $\text{ATL}^+$  and ATL can then be defined as follows, where

$l \in \text{Prop} \cup \{\neg p \mid p \in \text{Prop}\}$  is a literal,  $\mathbb{A}$  is a fixed finite set of agents and  $A \subseteq \mathbb{A}$ :

$$\text{State formulae: } \varphi := l \mid (\varphi \vee \varphi) \mid (\varphi \wedge \varphi) \mid \langle\langle A \rangle\rangle \Phi \mid \llbracket A \rrbracket \Phi \quad (1)$$

$$\text{ATL}^*\text{-path formulae: } \Phi := \varphi \mid \bigcirc \Phi \mid \square \Phi \mid (\Phi \mathcal{U} \Phi) \mid (\Phi \vee \Phi) \mid (\Phi \wedge \Phi) \quad (2)$$

$$\text{ATL}^+\text{-path formulae: } \Phi := \varphi \mid \bigcirc \varphi \mid \square \varphi \mid (\varphi \mathcal{U} \varphi) \mid (\Phi \vee \Phi) \mid (\Phi \wedge \Phi) \quad (3)$$

$$\text{ATL-path formulae: } \Phi := \bigcirc \varphi \mid \square \varphi \mid (\varphi \mathcal{U} \varphi) \quad (4)$$

Note that the state formulae have the same definition but define different sets in all 3 cases. To keep the notation lighter, we will list the members of the set  $A$  in  $\langle\langle A \rangle\rangle$  without using  $\{\}$ . When the length of a formula is measured,  $A$  will be assumed given by a bit vector. Parentheses will be omitted whenever safe, but they will be important when conjunctions and disjunctions are composed.

Hereafter, we use  $\varphi, \psi, \eta$  to denote arbitrary state formulae and  $\Phi, \Psi$  to denote path formulae. By an  $\text{ATL}^+$  formula we will mean by default a *state* formula of  $\text{ATL}^+$ ; likewise for  $\text{ATL}$ . We define  $\top := p \vee \neg p$ ,  $\perp := \neg \top$  and the temporal operators *Sometime*  $\Diamond$  by  $\Diamond \varphi := \top \mathcal{U} \varphi$  and *Release*  $\mathcal{R}$  by  $\psi \mathcal{R} \varphi := \square \varphi \vee \varphi \mathcal{U} (\varphi \wedge \psi)$ . Note, that  $\langle\langle A \rangle\rangle \psi \mathcal{R} \varphi$  and  $\llbracket A \rrbracket \psi \mathcal{R} \varphi$  are  $\text{ATL}^+$  state formulae.

$\text{CTL}^*$  can be regarded as the fragment of  $\text{ATL}^*$  where  $\langle\langle \emptyset \rangle\rangle$  represents the path quantifier  $\forall$  and  $\langle\langle \mathbb{A} \rangle\rangle$  represents  $\exists$ . The semantics of  $\text{ATL}^*$  (inherited by  $\text{ATL}^+$ ) is defined in a given CGM  $\mathcal{M}$ , state  $s \in \mathcal{M}$  and a path  $\lambda$  in  $\mathcal{M}$  just like the semantics of  $\text{CTL}^*$ , with the added clauses for the strategic quantifiers:

- $\mathcal{M}, s \models p$ , for any proposition  $p \in \text{Prop}$ , iff  $p \in L(s)$ .
- $\mathcal{M}, s \models \neg p$  iff  $\mathcal{M}, s \not\models p$ .
- $\mathcal{M}, s \models \varphi \wedge \psi$  iff  $\mathcal{M}, s \models \varphi$  and  $\mathcal{M}, s \models \psi$ .
- $\mathcal{M}, s \models \varphi \vee \psi$  iff  $\mathcal{M}, s \models \varphi$  or  $\mathcal{M}, s \models \psi$ .
- $\mathcal{M}, s \models \langle\langle A \rangle\rangle \Phi$  iff there exists an  $A$ -strategy  $F_A$  such that, for all computations  $\lambda$  consistent with  $F_A$ ,  $\mathcal{M}, \lambda \models \Phi$ .
- $\mathcal{M}, s \models \llbracket A \rrbracket \Phi$  iff there exists an  $A$ -co-strategy  $F_A^c$  such that, for all computations  $\lambda$  consistent with  $F_A^c$ ,  $\mathcal{M}, \lambda \models \Phi$ .
- $\mathcal{M}, \lambda \models \varphi$  iff  $\mathcal{M}, \lambda_0 \models \varphi$ .
- $\mathcal{M}, \lambda \models \bigcirc \varphi$  iff  $\mathcal{M}, \lambda_{\geq 1} \models \varphi$ .
- $\mathcal{M}, \lambda \models \square \varphi$  iff for all positions  $i \geq 0$ ,  $\mathcal{M}, \lambda_{\geq i} \models \varphi$ .
- $\mathcal{M}, \lambda \models \varphi \mathcal{U} \psi$  iff there exists a position  $i \geq 0$  where  $\mathcal{M}, \lambda_{\geq i} \models \psi$  and for all positions  $0 \leq j < i$ ,  $\mathcal{M}, \lambda_{\geq j} \models \varphi$ .
- $\mathcal{M}, \lambda \models \Phi \wedge \Psi$  iff  $\mathcal{M}, \lambda \models \Phi$  and  $\mathcal{M}, \lambda \models \Psi$ .
- $\mathcal{M}, \lambda \models \Phi \vee \Psi$  iff  $\mathcal{M}, \lambda \models \Phi$  or  $\mathcal{M}, \lambda \models \Psi$ .

Valid, satisfiable and equivalent formulae in  $\text{ATL}^*$  are defined as usual. Here are some important equivalences in LTL [7] and in  $\text{ATL}^*$  [1, 8], used further:

- $\Box\Psi \equiv \Psi \wedge \bigcirc\Box\Psi$ ;  $\Phi\mathcal{U}\Psi \equiv \Psi \vee (\Phi \wedge \bigcirc(\Phi\mathcal{U}\Psi))$ ;
- $\langle\langle A \rangle\rangle\Box\Psi \equiv \Psi \wedge \langle\langle A \rangle\rangle\bigcirc\langle\langle A \rangle\rangle\Box\Psi$ ;  $\langle\langle A \rangle\rangle\Phi\mathcal{U}\Psi \equiv \Psi \vee (\Phi \wedge \langle\langle A \rangle\rangle\bigcirc\langle\langle A \rangle\rangle\Phi\mathcal{U}\Psi)$ ;
- $\llbracket A \rrbracket\Box\Psi \equiv \Psi \wedge \llbracket A \rrbracket\bigcirc\llbracket A \rrbracket\Box\Psi$ ;  $\llbracket A \rrbracket\Phi\mathcal{U}\Psi \equiv \Psi \vee (\Phi \wedge \llbracket A \rrbracket\bigcirc\llbracket A \rrbracket\Phi\mathcal{U}\Psi)$ ;
- $\llbracket A \rrbracket\bigcirc\varphi \equiv \neg\langle\langle A \rangle\rangle\bigcirc\neg\varphi \equiv \langle\langle \emptyset \rangle\rangle\bigcirc\varphi$ ;  $\langle\langle A \rangle\rangle\langle\langle B \rangle\rangle\Phi \equiv \langle\langle B \rangle\rangle\Phi$ ;
- For every state formula  $\varphi$ :  $\langle\langle A \rangle\rangle(\varphi \wedge \Psi) \equiv \varphi \wedge \langle\langle A \rangle\rangle\Psi$ ,  $\langle\langle A \rangle\rangle(\varphi \vee \Psi) \equiv \varphi \vee \langle\langle A \rangle\rangle\Psi$ .

*Remark 2.1.* It is known [1] that, when restricted to ATL formulae, the semantics above (based on perfect-recall strategies) is equivalent to the semantics based on positional (or memoryless) strategies, where the prescribed actions only depend on the current state, not on the whole history. This is no longer the case for  $\text{ATL}^+$ . For example, the formula  $\langle\langle 1 \rangle\rangle\Diamond(p \wedge \langle\langle 1 \rangle\rangle\Diamond q) \rightarrow \langle\langle 1 \rangle\rangle(\Diamond p \wedge \Diamond q)$  in a 2-agents language is valid in the semantics with perfect-recall strategies (which can be freely composed) but not in the semantics with positional strategies (which cannot be freely composed). Indeed, in the concurrent game model of Figure 1, the antecedent of the above implication, namely  $\langle\langle 1 \rangle\rangle\Diamond(p \wedge \langle\langle 1 \rangle\rangle\Diamond q)$ , is true at state  $s_0$  no matter what strategy – perfect-recall or positional – is considered, whereas the consequent, namely  $\langle\langle 1 \rangle\rangle(\Diamond p \wedge \Diamond q)$ , is true at  $s_0$  only with respect to perfect-recall strategies. To be more precise, with respect to the state  $S_0$  only two cases of memoryless strategy  $F$  for player 1 are possible:  $F(S_0) = a$  and  $F(S_0) = b$ . Since the strategy is positional, these actions would be applied every time the play reaches  $S_0$ , and neither of them guarantees that the play will eventually visit both a state satisfying  $p$  and a state satisfying  $q$ . On the other hand, a perfect-recall strategy  $F$  such that  $F(S_0) = a$  and  $F(S_0S_1S_0) = b$  guarantees the satisfaction of the objective  $\Diamond p \wedge \Diamond q$ .

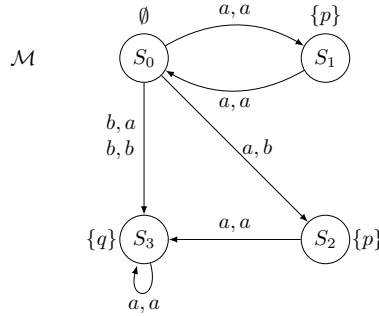


Figure 1: A CGM

Here we assume that the semantics is based on perfect-recall strategies.

The (*constructive*) *satisfiability decision problem* for  $\text{ATL}^+$  is defined as follows:

Given a state formula  $\varphi$  in  $\text{ATL}^+$ , does there exist a CGM  $\mathcal{M}$  and a state  $s$  in  $\mathcal{M}$  such that  $\mathcal{M}, s \models \varphi$ ? If so, construct such a satisfying pair  $(\mathcal{M}, s)$ .

*Remark 2.2.* There are three variants of the satisfiability problem: *tight*, where it is assumed that all agents in the model are mentioned in the formula, *loose* where just one additional agent, not mentioned in the formula is allowed in the model, and *general*, where any number of additional agents, not mentioned in the formula, are allowed in the model. These variants are really different, but the general satisfiability is immediately reducible to the loose satisfiability, by adding just one extra agent  $a$  to the language. Furthermore, this extra agent can be easily added superfluously to the formula, e.g., by adding a conjunct  $\langle\langle a \rangle\rangle \bigcirc \top$ , thus reducing loose to tight satisfiability. So, hereafter we only consider the tight satisfiability version. For further details and discussion on this issue, see e.g., [13, 9].

### 3 Decomposition and closure of $\text{ATL}^+$ formulae

We partition the set of  $\text{ATL}^+$  formulae into *primitive* and *non-primitive* formulae. The primitive formulae are  $\top, \perp$ , the literals and all  $\text{ATL}^+$  *successor formulae*, of the form  $\langle\langle A \rangle\rangle \psi$  or  $\llbracket A' \rrbracket \psi$ , where  $A \subseteq \mathbb{A}$  and  $A' \subset \mathbb{A}$ , each with *successor component*  $\psi$ . The non-primitive formulae are classified as  $\alpha$ -,  $\beta$ - and  $\gamma$ -formulae. An  $\alpha$ -formula in our syntax is a conjunction  $\varphi \wedge \psi$  with (conjunctive)  $\alpha$ -components  $\varphi$  and  $\psi$ , plus the formulae of the form  $\llbracket \mathbb{A} \rrbracket \psi$  whose  $\alpha$ -components are both  $\langle\langle \emptyset \rangle\rangle \psi$ ; a  $\beta$ -formula is a disjunction  $\varphi \vee \psi$  with (disjunctive)  $\beta$ -components  $\varphi$  and  $\psi$ . The rest of the non-primitive formulae are classified as  $\gamma$ -formulae. That is, a  $\gamma$ -formula is one of the form  $\llbracket A \rrbracket \Phi$  or  $\langle\langle A \rangle\rangle \Phi$ , where  $\Phi$  is an  $\text{ATL}^+$  path formula whose main operator is not  $\bigcirc$  and  $A \neq \mathbb{A}$ .

The need of introducing the new category of  $\gamma$ -formulae, w.r.t. the partition of non-primitive formulae into  $\alpha$ - and  $\beta$ - classes done in [9] is the following. In ATL each strategic quantifier is necessarily followed by a temporal operator, and, for instance  $\langle\langle A \rangle\rangle \Box \varphi$  can be seen as an  $\alpha$ -formula while  $\langle\langle A \rangle\rangle \varphi \mathcal{U} \psi$  can be seen as a  $\beta$ -formula. However, typical state formulae in  $\text{ATL}^+$  have the form  $\langle\langle A \rangle\rangle (\Phi_1 \vee \Phi_2)$ ,  $\langle\langle A \rangle\rangle (\Phi_1 \wedge \Phi_2)$ ,  $\llbracket A \rrbracket (\Phi_1 \wedge \Phi_2)$ ,  $\llbracket A \rrbracket (\Phi_1 \vee \Phi_2)$ . Now, these four types of formulae cannot reasonably be classified as  $\alpha$ - or  $\beta$ - formulae. Note, in particular, that the strategic quantifier  $\langle\langle A \rangle\rangle$  in general distributes neither on  $\vee$  nor on  $\wedge$  and the same applies to  $\llbracket A \rrbracket$ . Thus, a new category of  $\gamma$ -formulae is created, containing also  $\langle\langle A \rangle\rangle \Box \varphi$  and  $\langle\langle A \rangle\rangle \varphi \mathcal{U} \psi$  as special cases, and needing a special analysis.

Thus  $\alpha$ - and  $\beta$ -formulae will be decomposed in the tableau as usual, while the case of  $\gamma$ -formulae  $\langle\langle A \rangle\rangle \Phi$  and  $\llbracket A \rrbracket \Phi$  is special and needs extra work, because their tableau decomposition will depend on the structure of  $\Phi$ .

#### 3.1 $\gamma$ -decomposition and $\gamma$ -components of $\gamma$ -formulae

We denote the set of  $\text{ATL}^+$  state formulae by  $\text{ATL}_s^+$  and the set of  $\text{ATL}^+$  path formulae by  $\text{ATL}_p^+$ . We will define a  $\gamma$ -decomposition function  $\text{dec} : \text{ATL}_p^+ \rightarrow \mathcal{P}(\text{ATL}_s^+ \times \text{ATL}_p^+)$  with the following intuitive meaning: for any  $\Phi \in \text{ATL}_p^+$  and pair  $\langle \psi, \Psi \rangle \in \text{dec}(\Phi)$ ,  $\psi$  is a state formula true at the current state and  $\Psi$  is

a path formula expressing what must be true at the next state of a possible play starting at the current state. Thus, the set  $\text{dec}(\Phi)$  is interpreted as a disjunction describing all possible ‘types of paths’ starting from the current state and satisfying  $\Phi$ .

We emphasize that, although the domain of  $\text{dec}$  is the whole set  $\text{ATL}_p^+$ ,  $\text{dec}$  will only be used to analyse  $\Phi$  in the contexts  $\langle\langle A \rangle\rangle\Phi$  and  $\llbracket A \rrbracket\Phi$  (where  $\Phi$  does not have  $\bigcirc$  as main connective), and, as we will see, its role is just auxiliary to the rewriting of the (always quantified)  $\gamma$ -formulae in a special form useful to obtain a key ty of our tableau calculus (see further Lemma 3.1).

*Base cases:*

★  $\text{dec}(\varphi) = \{\langle\varphi, \top\rangle\}$ ,  $\text{dec}(\bigcirc\varphi) = \{\langle\top, \varphi\rangle\}$  for any  $\text{ATL}^+$  state formula  $\varphi$ .

The other base cases derive from the well-known LTL equivalences listed in 2.2:

★  $\text{dec}(\Box\varphi) = \{\langle\varphi, \Box\varphi\rangle\}$   
 ★  $\text{dec}(\varphi\mathcal{U}\psi) = \{\langle\varphi, \varphi\mathcal{U}\psi\rangle, \langle\psi, \top\rangle\}$ .

*Recursive steps:*

★  $\text{dec}(\Phi_1 \wedge \Phi_2) = \text{dec}(\Phi_1) \otimes \text{dec}(\Phi_2)$ , where

$\text{dec}(\Phi_1) \otimes \text{dec}(\Phi_2) := \{\langle\psi_i \wedge \psi_j, \Psi_i \wedge \Psi_j\rangle \mid \langle\psi_i, \Psi_i\rangle \in \text{dec}(\Phi_1), \langle\psi_j, \Psi_j\rangle \in \text{dec}(\Phi_2)\}$ .

★  $\text{dec}(\Phi_1 \vee \Phi_2) = \text{dec}(\Phi_1) \cup \text{dec}(\Phi_2) \cup (\text{dec}(\Phi_1) \oplus \text{dec}(\Phi_2))$ , where

$\text{dec}(\Phi_1) \oplus \text{dec}(\Phi_2) := \{\langle\psi_i \wedge \psi_j, \Psi_i \vee \Psi_j\rangle \mid \langle\psi_i, \Psi_i\rangle \in \text{dec}(\Phi_1), \langle\psi_j, \Psi_j\rangle \in \text{dec}(\Phi_2), \Psi_i \neq \top, \Psi_j \neq \top\}$ .

Note that the operations  $\otimes$  and  $\oplus$  are associative, up to logical equivalence.

The conjunctive case should be clear: every path satisfying  $\Phi_1 \wedge \Phi_2$  combines a type of path satisfying  $\Phi_1$  with a type of path satisfying  $\Phi_2$ . To understand the disjunctive case, note that, as it will be seen in Section 4, the construction of the tableau is step-by-step. Therefore, for a given prestate under construction, when we have a formula of the form  $\langle\langle A \rangle\rangle(\Phi_1 \vee \Phi_2)$ , where, for instance  $\Phi_1 = \Box\varphi_1$  and  $\Phi_2 = \Box\varphi_2$ , we do not know in advance which of  $\Box\varphi_1$  or  $\Box\varphi_2$  would be completed; so it is important to keep both possibilities at the current state, if possible. This idea is expressed by the use of  $\text{dec}(\Phi_1) \oplus \text{dec}(\Phi_2)$  in the above union, where we keep both disjuncts true at the present state and delay the choice. This is why the state formulae  $\psi_i$  and  $\psi_j$  are connected by  $\wedge$  but the path formulae  $\Psi_i$  and  $\Psi_j$  are connected by  $\vee$ . Moreover, the  $\oplus$  operation avoids the construction of a pair  $\langle\psi_i \wedge \psi_j, \Psi_i \vee \Psi_j\rangle$  where either  $\Psi_i$  or  $\Psi_j$  is  $\top$ , because that case would already be included in  $\text{dec}(\Phi_1)$  or in  $\text{dec}(\Phi_2)$ . The three cases for paths satisfying the disjunction  $\Phi_1 \vee \Phi_2$  can be illustrated by the picture in Figure 2.

Now, let  $\zeta = \langle\langle A \rangle\rangle\Phi$  or  $\zeta = \llbracket A \rrbracket\Phi$  be a  $\gamma$ -formula to be decomposed. Each pair  $\langle\psi, \Psi\rangle \in \text{dec}(\Phi)$  is then converted to a  $\gamma$ -component  $\gamma(\psi, \Psi)$  as follows:

$$\gamma(\psi, \Psi) = \psi \quad \text{if } \Psi = \top \quad (5)$$

$$\gamma(\psi, \Psi) = \psi \wedge \langle\langle A \rangle\rangle\bigcirc\langle\langle A \rangle\rangle\Psi \quad \text{if } \zeta \text{ is of the form } \langle\langle A \rangle\rangle\Phi, \quad (6)$$

$$\gamma(\psi, \Psi) = \psi \wedge \llbracket A \rrbracket\bigcirc\llbracket A \rrbracket\Psi \quad \text{if } \zeta \text{ is of the form } \llbracket A \rrbracket\Phi \quad (7)$$



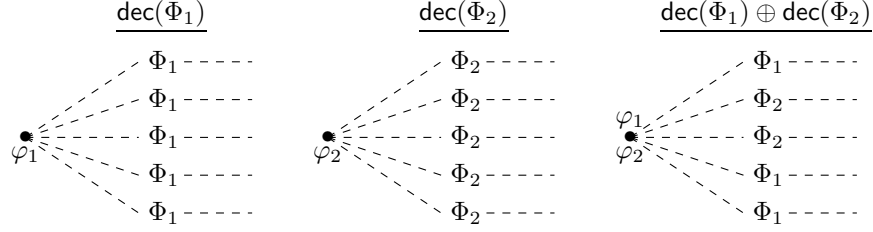


Figure 2: The three cases for disjunctive path objectives in a  $\gamma$ -formula.

Thus, the role of **dec** is to associate with any  $\gamma$ -formula  $\zeta$  a set of formulae that are simpler in some precise sense, viz. its  $\gamma$ -components, so that  $\zeta$  is equivalent to the disjunction of its  $\gamma$ -components. This key property is item 3 of the next lemma (the first two items being just auxiliary claims), and it is the core distinction between the proposed calculus for  $\text{ATL}^+$  in this work and the tableau calculus for  $\text{ATL}$  in [9].

**Lemma 3.1.** *For any  $\gamma$ -formula  $\Theta = \langle\langle A \rangle\rangle\Phi$  or  $\Theta = \llbracket A \rrbracket\Phi$  of  $\text{ATL}^+$ , the following properties hold:*

1.  $\Phi \equiv \bigvee \{ \psi \wedge \bigcirc \Psi \mid \langle \psi, \Psi \rangle \in \text{dec}(\Phi) \}$ .
2.  $\langle\langle A \rangle\rangle\Phi \equiv \bigvee \{ \langle\langle A \rangle\rangle(\psi \wedge \bigcirc \Psi) \mid \langle \psi, \Psi \rangle \in \text{dec}(\Phi) \}$ , and respectively,  
 $\llbracket A \rrbracket\Phi \equiv \bigvee \{ \llbracket A \rrbracket(\psi \wedge \bigcirc \Psi) \mid \langle \psi, \Psi \rangle \in \text{dec}(\Phi) \}$ .
3.  $\Theta \equiv \bigvee \{ \gamma(\psi, \Psi) \mid \langle \psi, \Psi \rangle \in \text{dec}(\Phi) \}$ .

*Proof.* Claim 1. We will prove the claim by induction on the path formula  $\Phi$ . It is equivalent to the following property  $P(\Phi)$ :

For every CGM  $\mathcal{M}$  and a play  $\lambda$  in it,  $\mathcal{M}, \lambda \models \Phi$  iff there exists  $\langle \psi, \Psi \rangle \in \text{dec}(\Phi)$  such that  $\mathcal{M}, \lambda_0 \models \psi$  and  $\mathcal{M}, \lambda_{\geq 1} \models \Psi$ .

The base cases are  $\Phi = \varphi$ ,  $\Phi = \bigcirc \varphi$ ,  $\Phi = \Box \varphi$  and  $\Phi = \varphi \mathcal{U} \psi$ . For each of these the property  $P(\Phi)$  follows immediately from the definitions of **dec** and  $\gamma$ -components and – for the latter two cases – the well-known fixed point LTL equivalences for the temporal operators, listed at the end of Section 2.2.

For the inductive steps there are two cases to consider:

Case 1:  $\Phi = \Phi_1 \wedge \Phi_2$ . We have that:

$\mathcal{M}, \lambda \models \Phi$  iff

$\mathcal{M}, \lambda \models \Phi_1$  and  $\mathcal{M}, \lambda \models \Phi_2$ , iff (by the induction hypothesis):

- (i) there is  $\langle \psi_1, \Psi_1 \rangle \in \text{dec}(\Phi_1)$ , such that  $\mathcal{M}, \lambda_0 \models \psi_1$  and  $\mathcal{M}, \lambda_{\geq 1} \models \Psi_1$ ,  
and
- (ii) there is  $\langle \psi_2, \Psi_2 \rangle \in \text{dec}(\Phi_2)$  such that  $\mathcal{M}, \lambda_0 \models \psi_2$  and  $\mathcal{M}, \lambda_{\geq 1} \models \Psi_2$ .

These two are the case iff

$\mathcal{M}, \lambda_0 \models \psi_1 \wedge \psi_2$  and  $\mathcal{M}, \lambda_{\geq 1} \models \Psi_1 \wedge \Psi_2$ , iff  
 $\mathcal{M}, \lambda_0 \models \psi$  and  $\mathcal{M}, \lambda_{\geq 1} \models \Psi$  where  $\psi = \psi_1 \wedge \psi_2$ ,  $\Psi = \Psi_1 \wedge \Psi_2$  and  $\langle \psi, \Psi \rangle \in \text{dec}(\Phi)$ . This completes the proof of  $P(\Phi)$  for  $\Phi = \Phi_1 \wedge \Phi_2$ .

Case 2:  $\Phi = \Phi_1 \vee \Phi_2$ . We have that  $\mathcal{M}, \lambda \models \Phi$  iff  $\mathcal{M}, \lambda \models \Phi_1$  or  $\mathcal{M}, \lambda \models \Phi_2$ . By inductive hypotheses for  $\Phi_1$  and  $\Phi_2$  and from the fact that  $\text{dec}(\Phi_1) \cup \text{dec}(\Phi_2) \subseteq \text{dec}(\Phi)$ , we obtain the direction from left to right in property  $P(\Phi)$ . For the converse direction, we only need to consider the case that does not follow directly from the inductive hypotheses for  $\Phi_1$  and  $\Phi_2$ , viz. when there exists  $\langle \psi, \Psi \rangle \in (\text{dec}(\Phi_1) \oplus \text{dec}(\Phi_2))$  such that  $\mathcal{M}, \lambda_0 \models \psi$  and  $\mathcal{M}, \lambda_{\geq 1} \models \Psi$ . In this case,  $\psi = \psi_1 \wedge \psi_2$  and  $\Psi = \Psi_1 \vee \Psi_2$  for some  $\langle \psi_1, \Psi_1 \rangle \in \text{dec}(\Phi_1)$  and  $\langle \psi_2, \Psi_2 \rangle \in \text{dec}(\Phi_2)$  such that  $\Psi_1 \neq \top, \Psi_2 \neq \top$ . Suppose  $\mathcal{M}, \lambda_{\geq 1} \models \Psi_1$ . Since we also have  $\mathcal{M}, \lambda_0 \models \psi_1$ , by the inductive hypothesis for  $\Phi_1$ , it follows that  $\mathcal{M}, \lambda \models \Phi_1$ , hence  $\mathcal{M}, \lambda \models \Phi$ . Likewise, when  $\mathcal{M}, \lambda_{\geq 1} \models \Psi_2$ .

Claim 2. We will consider the case of  $\Theta = \langle\langle A \rangle\rangle \Phi$ ; the case of  $\llbracket A \rrbracket \Phi$  is analogous. The implication from right to left of the claimed equivalence follows from Claim 1 and the monotonicity of  $\langle\langle A \rangle\rangle$  (in sense that if  $\Psi \models \Phi$  then  $\langle\langle A \rangle\rangle \Psi \models \langle\langle A \rangle\rangle \Phi$ ). For the converse direction, first recall that every  $\text{ATL}^+$  path formula  $\Xi$  is a positive Boolean combination of sub-formulae of the types  $\varphi, \bigcirc \varphi, \square \varphi, \varphi \mathcal{U} \psi$  where  $\varphi, \psi$  are  $\text{ATL}^+$  state formulae. Let the set of these sub-formulae of  $\Xi$  be  $S(\Xi)$ . Now, we introduce some ad hoc notation for special sets of formulae in  $S(\Xi)$  and their sub-formulae:

- $L(\Xi)$  is the set of all state formulae in  $S(\Xi)$ ;
- $N(\Xi) := \{\varphi \mid \bigcirc \varphi \in S(\Xi)\}$ ;
- $B(\Xi) := \{\varphi \mid \square \varphi \in S(\Xi)\}$ ;
- $U(\Xi) := \{\varphi \mathcal{U} \psi \mid \varphi \mathcal{U} \psi \in S(\Xi)\}$ ;
- $U_1(\Xi) := \{\varphi \mid \varphi \mathcal{U} \psi \in S(\Xi)\}$ ;
- $U_2(\Xi) := \{\psi \mid \varphi \mathcal{U} \psi \in S(\Xi)\}$ ;

Without loss of generality we can assume that  $\Phi$  is in a DNF over the set of formulae in  $S(\Phi)$ , i.e.  $\Phi = \Phi_1 \vee \dots \vee \Phi_m$ , where each  $\Phi_i$  is a conjunction of formulae from  $S(\Phi)$ .

Now, to prove the implication from left to right, take any CGM  $\mathcal{M}$  and state  $s$  in it, such that  $\mathcal{M}, s \models \langle\langle A \rangle\rangle \Phi$ . Take and fix any collective strategy  $F_A$  of  $A$  such that  $\mathcal{M}, \lambda \models \Phi$  for every play  $\lambda$  starting at  $s$  and consistent with  $F_A$ . We denote that set of plays by  $\text{OutPath}(s, F_A)$ . Then for every play  $\lambda \in \text{out}_{\mathcal{M}}(s, \sigma_A)$  we have that  $\mathcal{M}, \lambda \models \Phi_i$  for some  $i = 1, \dots, m$ . Without restriction of generality we can assume that the set of  $\Phi_i$ 's for which there is a  $\lambda \in \text{out}_{\mathcal{M}}(s, \sigma_A)$  such that  $\mathcal{M}, \lambda \models \Phi_i$  is  $\{\Phi_1, \dots, \Phi_n\}$  for some  $n \leq m$ .

Let  $\Phi_i$  be any of these. We will associate with it a pair  $\langle \psi_i, \Psi_i \rangle \in \text{dec}(\Phi_i)$  as follows. First, note that all formulae from  $L(\Phi_i)$  and  $B(\Phi_i)$  are true at  $s$ .

Further, let  $E_i(s)$  be the subset of those formulae from  $U_2(\Phi_i)$  which are true at  $s$  in  $\mathcal{M}$ . Thus, for every play  $\lambda \in \text{OutPath}(s, F_A)$  satisfying  $\Phi_i$  the following hold:

- i)  $\mathcal{M}, \lambda \models \varphi$  for each  $\varphi \in L(\Phi_i)$ .
- ii)  $\mathcal{M}, \lambda \models \bigcirc \varphi$  for each  $\bigcirc \varphi \in S(\Phi_i)$ .
- iii)  $\mathcal{M}, \lambda \models \varphi \wedge \bigcirc \square \varphi$  for each  $\square \varphi \in S(\Phi_i)$ .
- iv)  $\mathcal{M}, \lambda \models \psi$  for each  $\psi \in E_i(s)$ .
- v)  $\mathcal{M}, \lambda \models \varphi \wedge \bigcirc \varphi \mathcal{U} \psi$  for each  $\psi \in U_2(\Phi_i) - E_i(s)$ .

Now, suppose  $\Phi_i = \Psi_{i1} \wedge \dots \wedge \Psi_{ik}$  for some  $\Psi_{i1} \wedge \dots \wedge \Psi_{ik} \in S(\Phi)$ . Then  $\text{dec}(\Phi_i) = \text{dec}(\Psi_{i1}) \otimes \dots \otimes \text{dec}(\Psi_{ik})$ . (Recall that the operations  $\otimes$  and  $\oplus$  are associative, up to logical equivalence, so there is no need to put parentheses.) Thus, for every  $\langle \psi, \Psi \rangle \in \text{dec}(\Phi_i)$ ,  $\psi$  is a conjunction of all formulae from  $L(\Phi_i) \cup B(\Phi_i)$  and, for every conjunct of  $\Phi_i$  of the type  $\varphi \mathcal{U} \psi$ , at least one of the respective formulae coming from  $U_1(\Phi_i)$  and  $U_2(\Phi_i)$ . We now select  $\langle \psi_i, \Psi_i \rangle \in \text{dec}(\Phi_i)$  to be the one where the conjuncts taken from  $U_2(\Phi_i)$  are exactly those in  $E_i(s)$ . Then we claim that for every play  $\lambda \in \text{OutPath}(s, F_A)$  satisfying  $\Phi_i$ , it is the case that  $\mathcal{M}, \lambda \models \psi_i \wedge \bigcirc \Psi_i$ . Indeed, this follows from the list of properties (i - v) above and from the definition of  $\text{dec}(\Psi_{i1}) \otimes \dots \otimes \text{dec}(\Psi_{ik})$ . Note further, that if  $\Psi_i$  above is  $\top$ , then  $\mathcal{M}, \lambda \models \psi_i \wedge \bigcirc \Psi_i$  for all paths  $\lambda$  starting at  $s$ , so we can assume without affecting what follows that no  $\Psi_i$  above is  $\top$ .

After having selected such a pair  $\langle \psi_i, \Psi_i \rangle \in \text{dec}(\Phi_i)$  for each  $\Phi_i \in \{\Phi_1, \dots, \Phi_n\}$ , we use these  $n$  pairs (or, those of them for which  $\Psi_i \neq \top$ ) to construct the pair  $\langle \psi, \Psi \rangle \in \text{dec}(\Phi_1) \oplus \dots \oplus \text{dec}(\Phi_n)$  such that  $\psi = \psi_1 \wedge \dots \wedge \psi_n$  and  $\Psi = \Psi_1 \vee \dots \vee \Psi_n$ .

Finally, we claim that, by virtue of the construction,  $\mathcal{M}, \lambda \models \psi \wedge \bigcirc \Psi$  for every play  $\lambda \in \text{OutPath}(s, F_A)$  satisfying  $\Phi$ . Therefore, the strategy  $\sigma_A$  is a witness of the truth of  $\mathcal{M}, s \models \langle\langle A \rangle\rangle(\psi \wedge \bigcirc \Psi)$ , hence  $\mathcal{M}, s \models \bigvee \{ \langle\langle A \rangle\rangle(\psi \wedge \bigcirc \Psi) \mid \langle \psi, \Psi \rangle \in \text{dec}(\Phi) \}$ . This completes the proof of the implication left-to-right of Claim 2.

Claim 3. This claim follows easily from Claim 2 by noting that:

- $\langle\langle A \rangle\rangle(\psi \wedge \bigcirc \square \Psi) \equiv \psi \wedge \langle\langle A \rangle\rangle \bigcirc \square \Psi \equiv \psi \wedge \langle\langle A \rangle\rangle \bigcirc \langle\langle A \rangle\rangle \square \Psi$ , because  $\psi$  is a state formula. Note that the second equivalence is due to the fact that the semantics of  $\langle\langle \rangle\rangle$  is based on perfect recall strategies, that can be composed. More precisely, it essentially assumes that any strategy at  $s$  ensuring that every successor satisfies  $\langle\langle A \rangle\rangle \square \Psi$  can be composed with the family of strategies, one for every such successor  $s'$  witnessing the truth of  $\langle\langle A \rangle\rangle \square \Psi$  on all plays starting at  $s'$ , into one perfect recall strategy that guarantees the truth of  $\bigcirc \square \Psi$  on all plays starting at  $s$ . (This, in general, cannot be done if only positional strategies are considered, as those applied at the different successors of  $s$  may interfere with each other.)

- Likewise,  $\llbracket A \rrbracket(\psi \wedge \bigcirc \square \Psi) \equiv \psi \wedge \llbracket A \rrbracket \bigcirc \square \Psi \equiv \psi \wedge \llbracket A \rrbracket \bigcirc \llbracket A \rrbracket \square \Psi$ .

Therefore, for each  $\langle \psi, \Psi \rangle \in \text{dec}(\Phi)$  the  $\gamma$ -component  $\gamma(\psi, \Psi)$  is equivalent to its respective disjunct on the right hand side of Claim 2.  $\square$   $\square$

**Example 3.1.** We will use two syntactically similar, yet different, running examples:

$$\theta = \langle\langle 1 \rangle\rangle(p\mathcal{U}q \vee \square q) \wedge \langle\langle 2 \rangle\rangle(\Diamond p \wedge \square \neg q)$$

and

$$\vartheta = \langle\langle 1 \rangle\rangle(p\mathcal{U}q \vee \square q) \wedge \llbracket 2 \rrbracket(\Diamond p \wedge \square \neg q).$$

First, we consider  $\theta$ . It is an  $\alpha$ -formula with conjunctive components  $\theta_1 = \langle\langle 1 \rangle\rangle(p\mathcal{U}q \vee \square q)$  and  $\theta_2 = \langle\langle 2 \rangle\rangle(\Diamond p \wedge \square \neg q)$ .

Further,  $\theta_1$  is a  $\gamma$ -formula of the form  $\langle\langle A \rangle\rangle \Phi$  where the main connective of  $\Phi$  is  $\vee$ . So,  $\text{dec}(\theta_1) = \text{dec}(p\mathcal{U}q) \cup \text{dec}(\square q) \cup (\text{dec}(p\mathcal{U}q) \oplus \text{dec}(\square q))$ , where  $\text{dec}(p\mathcal{U}q) = \{\langle p, p\mathcal{U}q \rangle, \langle q, \top \rangle\}$  and  $\text{dec}(\square q) = \{\langle q, \square q \rangle\}$ .

Thus,  $\text{dec}(\theta_1) = \{\langle p, p\mathcal{U}q \rangle, \langle q, \top \rangle, \langle q, \square q \rangle, \langle p \wedge q, p\mathcal{U}q \vee \square q \rangle\}$ , hence  $\theta_1 \equiv (p \wedge \langle\langle 1 \rangle\rangle \bigcirc \langle\langle 1 \rangle\rangle p\mathcal{U}q) \vee q \vee (q \wedge \langle\langle 1 \rangle\rangle \bigcirc \langle\langle 1 \rangle\rangle \square q) \vee (p \wedge q \wedge \langle\langle 1 \rangle\rangle \bigcirc \langle\langle 1 \rangle\rangle (p\mathcal{U}q \vee \square q))$ .

Likewise,  $\theta_2$  is a  $\gamma$ -formula of the form  $\langle\langle A \rangle\rangle \Phi$  and the main connective of  $\Phi$  is  $\wedge$ . So  $\text{dec}(\theta_2) = \text{dec}(\Diamond p) \otimes \text{dec}(\square \neg q)$ , with  $\text{dec}(\Diamond p) = \{\langle T, \Diamond p \rangle, \langle p, T \rangle\}$  and  $\text{dec}(\square \neg q) = \{\langle \neg q, \square \neg q \rangle\}$ .

Thus,  $\text{dec}(\theta_2) = \{\langle \top \wedge \neg q, \Diamond p \wedge \square \neg q \rangle, \langle p \wedge \neg q, \top \wedge \square \neg q \rangle\}$   
 $= \{\langle \neg q, \Diamond p \wedge \square \neg q \rangle, \langle p \wedge \neg q, \square \neg q \rangle\}$  and

$$\theta_2 \equiv (\neg q \wedge \langle\langle 2 \rangle\rangle \bigcirc \langle\langle 2 \rangle\rangle (\Diamond p \wedge \square \neg q)) \vee (p \wedge \neg q \wedge \langle\langle 2 \rangle\rangle \bigcirc \langle\langle 2 \rangle\rangle \square \neg q).$$

For  $\vartheta$ , the  $\gamma$ -decomposition is similar, we only replace  $\langle\langle 2 \rangle\rangle$  by  $\llbracket 2 \rrbracket$ . Thus, we obtain

$$\vartheta_1 \equiv (p \wedge \langle\langle 1 \rangle\rangle \bigcirc \langle\langle 1 \rangle\rangle p\mathcal{U}q) \vee (q) \vee (q \wedge \langle\langle 1 \rangle\rangle \bigcirc \langle\langle 1 \rangle\rangle \square q) \vee (p \wedge q \wedge \langle\langle 1 \rangle\rangle \bigcirc \langle\langle 1 \rangle\rangle (p\mathcal{U}q \vee \square q))$$

and

$$\vartheta_2 \equiv (\neg q \wedge \llbracket 2 \rrbracket \bigcirc \llbracket 2 \rrbracket (\Diamond p \wedge \square \neg q)) \vee (p \wedge \neg q \wedge \llbracket 2 \rrbracket \bigcirc \llbracket 2 \rrbracket \square \neg q).$$

The *closure*  $cl(\psi)$  of an  $\text{ATL}^+$  state formula  $\psi$  is the least set of  $\text{ATL}^+$  formulae such that  $\psi, \top, \perp \in cl(\psi)$  and  $cl(\psi)$  is closed under taking of successor-,  $\alpha$ -,  $\beta$ - and  $\gamma$ -components. For any set of state formulae  $\Gamma$  we define

$$cl(\Gamma) := \bigcup \{cl(\psi) \mid \psi \in \Gamma\}.$$

We denote by  $|\psi|$  the length of  $\psi$  and by  $\|\Gamma\|$  the cardinality of  $\Gamma$ .

**Example 3.2.** The construction of the closure of the formula  $\theta$  from Example 3.1 is given in Figure 3. Each node of the tree represents an element of the closure. Children of an interior node are respective components of the parent formula, according to the definition of closure.

The closure of  $\vartheta$  is similar to the one of  $\theta$  except that every  $\llbracket 2 \rrbracket$  is replaced by  $\langle\langle 2 \rangle\rangle$ .

**Lemma 3.2.** For any  $\text{ATL}^+$  state formula  $\varphi$ ,  $\|cl(\varphi)\| < 2^{|\varphi|^2}$ .

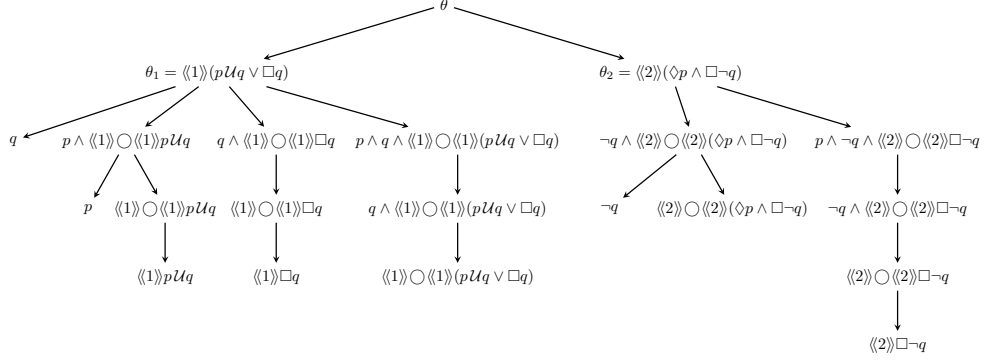


Figure 3: Closure of the formula  $\theta = \langle\langle 1 \rangle\rangle(p\mathcal{U}q \vee \Box q) \wedge \langle\langle 2 \rangle\rangle(\Diamond p \wedge \Box \neg q)$

*Proof.* Every formula in  $cl(\varphi)$  has length less than  $2|\varphi|$  and is built from symbols in  $\varphi$ , so there can be at most  $|\varphi|^{2|\varphi|} = 2^{2|\varphi|\log_2|\varphi|} < 2^{|\varphi|^2}$  such formulae.  $\square$   $\square$

The estimate above is rather crude, but  $\|cl(\varphi)\|$  can reach size exponential in  $|\varphi|$ . Indeed, consider the formulae  $\phi_k = \langle\langle 1 \rangle\rangle(p_1\mathcal{U}q_1 \wedge (p_2\mathcal{U}q_2 \wedge (\dots \wedge p_k\mathcal{U}q_k) \dots))$  for  $k = 1, 2, \dots$  and distinct  $p_1, q_1, \dots, p_k, q_k, \dots \in \text{Prop}$ . Then  $|\phi_k| = O(k)$ , while the number of different  $\gamma$ -components of  $\phi_k$  is  $2^k$ , hence  $\|cl(\phi_k)\| > 2^k$ .

### 3.2 Full expansions of sets of $\text{ATL}^+$ formulae

As part of the tableau construction we will need a procedure that, for any given finite set of  $\text{ATL}^+$  state formulae  $\Gamma$ , produces all “full expansions” (called in [9] “downward saturated extensions”; see Remark 3.1) defined below.

**Definition 3.1.** Let  $\Gamma, \Delta$  be sets of  $\text{ATL}^+$  state formulae and  $\Gamma \subseteq \Delta \subseteq cl(\Gamma)$ .

1.  $\Delta$  is *patently inconsistent* if it contains  $\perp$  or a pair of formulae  $\varphi$  and  $\neg\varphi$ .
2.  $\Delta$  is a *full expansion* of  $\Gamma$  if it is not patently inconsistent and satisfies the following closure conditions:
  - if  $\varphi \wedge \psi \in \Delta$  then  $\varphi \in \Delta$  and  $\psi \in \Delta$ ;
  - if  $\varphi \vee \psi \in \Delta$  then  $\varphi \in \Delta$  or  $\psi \in \Delta$ ;
  - if  $\varphi \in \Delta$  is a  $\gamma$ -formula, then at least one  $\gamma$ -component of  $\varphi$  is in  $\Delta$  and exactly one of these  $\gamma$ -components in  $\Delta$ , denoted  $\gamma(\varphi, \Delta)$ , is designated as *the  $\gamma$ -component in  $\Delta$  linked to the  $\gamma$ -formula  $\varphi$* , as explained below.

The family of all full expansions of  $\Gamma$  will be denoted by  $FE(\Gamma)$ . It can be constructed by a simple iterative procedure that starts with  $\{\Gamma\}$  and repeatedly, until saturation, takes a set  $X$  from the currently constructed family, selects a formula  $\varphi \in X$  and applies the closure rule above corresponding to its type.

Clearly, this procedure terminates on every finite input set of formulae  $\Gamma$  and produces a family of at most  $2^{\|\text{cl}(\Gamma)\|}$  sets. Furthermore, due to Lemma 3.1, we have the following:

**Proposition 3.1.** *For any finite set of  $\text{ATL}^+$  state formulae  $\Gamma$ :*

$$\bigwedge \Gamma \equiv \bigvee \left\{ \bigwedge \Delta \mid \Delta \in FE(\Gamma) \right\}.$$

*Proof.* Lemma 3.1 implies that every set extension step, described above, applied to a family of sets  $\mathcal{F}$  preserves the formula  $\bigvee \{ \bigwedge \Delta \mid \Delta \in \mathcal{F} \}$  up to logical equivalence. At the beginning, that formula is  $\bigwedge \Gamma$ .  $\square$   $\square$

*Remark 3.1.* Instead of full expansions, the tableau construction in [9] uses 'minimal downward saturated extensions', where 'downward saturated extension' corresponds to 'full expansion'. The minimality condition means that if one full expansion is contained in another one, then it is omitted. This could be problematic, as sometimes non-minimal full expansions may be needed. For instance, if  $\Gamma = \{ \langle\langle 1 \rangle\rangle(p\mathcal{U}q), p \wedge \langle\langle 1 \rangle\rangle \bigcirc \langle\langle 1 \rangle\rangle(p\mathcal{U}q) \}$  then  $FE(\Gamma) = \{ \{ \langle\langle 1 \rangle\rangle(p\mathcal{U}q), p \wedge \langle\langle 1 \rangle\rangle \bigcirc \langle\langle 1 \rangle\rangle(p\mathcal{U}q), p, \langle\langle 1 \rangle\rangle \bigcirc \langle\langle 1 \rangle\rangle(p\mathcal{U}q) \}, \{ q, \langle\langle 1 \rangle\rangle(p\mathcal{U}q), p \wedge \langle\langle 1 \rangle\rangle \bigcirc \langle\langle 1 \rangle\rangle(p\mathcal{U}q), p, \langle\langle 1 \rangle\rangle \bigcirc \langle\langle 1 \rangle\rangle(p\mathcal{U}q) \} \}$ .

Although the second full expansion contains the first one, we might have to consider both alternatives in the tableau where  $\Gamma$  is only part of the label of a state, for the sake of satisfying an eventuality of the type  $\varphi\mathcal{U}\psi$ . However, we have no concrete example showing that such situation may occur indeed.

## 4 Tableau-based decision procedure for $\text{ATL}^+$

The tableau procedure consists of three major phases: *pretableau construction*, *prestate elimination* and *state elimination*. Given an input formula  $\eta$ , it essentially constructs a (non-deterministic) CGS which is state-labelled by the closure set of the input formula  $\text{cl}(\eta)$ , i.e., a directed graph  $\mathcal{T}^\eta$  (called a *tableau*) where each node is labelled by a subset of  $\text{cl}(\eta)$  (see Def. 2.2(1)), and directed edges between nodes relating them to successor nodes.

The pretableau construction phase produces the so-called *pretableau*  $\mathcal{P}^\eta$  for the input formula  $\eta$ , with two kinds of nodes: *states* and *prestates*. States are fully expanded sets, meant to represent states of a CGM, while prestates can be any finite sets of formulae from  $\text{cl}(\eta)$  and only play a temporary role in the construction of  $\mathcal{P}^\eta$ . States and prestates are labelled uniquely, so they can be identified with their labels. The prestate elimination phase creates a smaller graph  $\mathcal{T}_0^\eta$  out of  $\mathcal{P}^\eta$ , called the *initial tableau for  $\eta$* , by eliminating all the prestates from  $\mathcal{P}^\eta$  and accordingly redirecting its edges. Finally, the state elimination phase removes, step-by-step, all the states (if any) that cannot be satisfied in a CGM, because they lack necessary successors or because they contain unrealized eventualities. Eventually, the elimination procedure produces a (possibly empty) subgraph  $\mathcal{T}^\eta$  of  $\mathcal{T}_0^\eta$ , called the *final tableau for  $\eta$* . If some state  $\Delta$  of  $\mathcal{T}^\eta$  contains  $\eta$ , the tableau procedure declares  $\eta$  satisfiable and a partly

defined CGM (called *Hintikka game structure*) satisfying  $\eta$  can be extracted from it by another procedure described in Section 6.2; otherwise it declares  $\eta$  unsatisfiable.

#### 4.1 Pretableau construction phase

The pretableau construction phase for an input formula  $\eta$  starts with an initial prestate (with label)  $\{\eta\}$  and consists of alternating application of two construction rules, until saturation: **(SR)**, expanding prestates into states, and **(Next)**, creating successor prestates from states. This phase closely resembles the corresponding one for the ATL tableaux in [9], with the only essential difference being the  $\gamma$ -decomposition of  $\gamma$ -formulae used here by the rule **(SR)**, which causes, as we will see, a possibly exponential blow-up of the size of the tableaux, and eventually of the entire worst-case time complexity, as compared to the ATL tableaux. Another (minor) difference with respect to [9] is in the formulation of both rules, because here we work with formulae in negation normal form.

**Rule (SR)** Given a prestate  $\Gamma$ , do the following:

1. For each full expansion  $\Delta$  of  $\Gamma$  add to the pretableau a state with label  $\Delta$ .
2. For each of the added states  $\Delta$ , if  $\Delta$  does not contain any formulae of the form  $\langle\langle A \rangle\rangle \bigcirc \varphi$  or  $\llbracket A \rrbracket \bigcirc \varphi$ , add the formula  $\langle\langle A \rangle\rangle \bigcirc \top$  to it;
3. For each state  $\Delta$  obtained at steps 1 and 2, link  $\Gamma$  to  $\Delta$  via a  $\Rightarrow$  edge;
4. If, however, the pretableau already contains a state  $\Delta'$  with label  $\Delta$ , do not create another copy of it but only link  $\Gamma$  to  $\Delta'$  via a  $\Rightarrow$  edge.

**Example 4.1.** For the formula  $\theta$  from Example 3.1 the initial prestate is

$$\Gamma_0^\theta = \{\langle\langle 1 \rangle\rangle(p\mathcal{U}q \vee \Box q) \wedge \langle\langle 2 \rangle\rangle(\Diamond p \wedge \Box \neg q)\}.$$

It has two full expansions:

$$\begin{aligned} \Delta_1 &= \{\theta, \theta_1, \theta_2, p, \neg q, \langle\langle 1 \rangle\rangle \bigcirc \langle\langle 1 \rangle\rangle p\mathcal{U}q, \langle\langle 2 \rangle\rangle \bigcirc \langle\langle 2 \rangle\rangle (\Diamond p \wedge \Box \neg q)\}, \text{ and} \\ \Delta_2 &= \{\theta, \theta_1, \theta_2, p, p \wedge \neg q, \neg q, \langle\langle 1 \rangle\rangle \bigcirc \langle\langle 1 \rangle\rangle p\mathcal{U}q, \langle\langle 2 \rangle\rangle \bigcirc \langle\langle 2 \rangle\rangle \Box \neg q\}. \end{aligned}$$

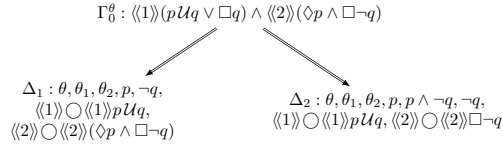


Figure 4: Application of the rule **(SR)** on the prestate  $\Gamma_0^\theta = \{\langle\langle 1 \rangle\rangle(p\mathcal{U}q \vee \Box q) \wedge \langle\langle 2 \rangle\rangle(\Diamond p \wedge \Box \neg q)\}$

Likewise, for the formula  $\vartheta$  the initial prestate is

$$\Gamma_0^\vartheta = \{\langle\langle 1 \rangle\rangle(p\mathcal{U}q \vee \Box q) \wedge \llbracket 2 \rrbracket(\Diamond p \wedge \Box \neg q)\}$$

and it has 2 full expansions:

$$\begin{aligned}\Delta_1 &= \{\vartheta, \vartheta_1, \vartheta_2, p, \neg q, \langle\langle 1 \rangle\rangle \circ \langle\langle 1 \rangle\rangle p\mathcal{U}q, \llbracket 2 \rrbracket \circ \llbracket 2 \rrbracket (\Diamond p \wedge \Box \neg q)\}, \text{ and} \\ \Delta_2 &= \{\vartheta, \vartheta_1, \vartheta_2, p, p \wedge \neg q, \neg q, \langle\langle 1 \rangle\rangle \circ \langle\langle 1 \rangle\rangle p\mathcal{U}q, \llbracket 2 \rrbracket \circ \llbracket 2 \rrbracket \Box \neg q\}.\end{aligned}$$

In the following, by *enforceable successor formula* we mean a formula of the form  $\langle\langle A \rangle\rangle \circ \psi$  and by *unavoidable successor formula* – one of the form  $\llbracket A' \rrbracket \circ \psi$ .

**Rule (Next)** Given a state  $\Delta$ , do the following, where  $\sigma$  is a shorthand for  $\sigma_{\mathbb{A}}$ :

1. List all primitive successor formulae of  $\Delta$  in such a way that all enforceable successor formulae precede all unavoidable ones where  $A \neq \mathbb{A}$ ; let the result be the list

$$\mathbb{L} = \langle\langle A_0 \rangle\rangle \circ \varphi_0, \dots, \langle\langle A_{m-1} \rangle\rangle \circ \varphi_{m-1}, \llbracket A'_0 \rrbracket \circ \psi_0, \dots, \llbracket A'_{l-1} \rrbracket \circ \psi_{l-1}$$

Let  $r_\Delta = m + l$ ; denote by  $D(\Delta)$  the set  $\{0, \dots, r_\Delta - 1\}^{|\mathbb{A}|}$ . Then, for every  $\sigma \in D(\Delta)$ , denote  $N(\sigma) := \{i \mid \sigma_i \geq m\}$ , where  $\sigma_i$  is the  $i$ th component of the tuple  $\sigma$ , and let  $\text{co}(\sigma) := [\sum_{i \in N(\sigma)} (\sigma_i - m)] \bmod l$ .

2. For each  $\sigma \in D(\Delta)$  create a prestate:

$$\begin{aligned}\Gamma_\sigma &= \{\varphi_p \mid \langle\langle A_p \rangle\rangle \circ \varphi_p \in \Delta \text{ and } \sigma_a = p \text{ for all } a \in A_p\} \\ &\cup \{\psi_q \mid \llbracket A'_q \rrbracket \circ \psi_q \in \Delta, \text{co}(\sigma) = q, \text{ and } \mathbb{A} - A'_q \subseteq N(\sigma)\}\end{aligned}$$

If  $\Gamma_\sigma$  is empty, add  $\top$  to it. Then connect  $\Delta$  to  $\Gamma_\sigma$  with  $\xrightarrow{\sigma}$ .

If, however,  $\Gamma_\sigma = \Gamma$  for some prestate  $\Gamma$  that has already been added to the pretableau, only connect  $\Delta$  to  $\Gamma$  with  $\xrightarrow{\sigma}$ .

*Remark 4.1.* Rule **(Next)** ensures that every prestate  $\Gamma$  of  $\Delta$ , that is every element of the finite set of prestates that are targets of  $\longrightarrow$  edges outgoing from  $\Delta$ , satisfies the following:

- if  $\{\langle\langle A_i \rangle\rangle \circ \varphi_i, \langle\langle A_j \rangle\rangle \circ \varphi_j\} \subseteq \Delta$  and  $\{\varphi_i, \varphi_j\} \subseteq \Gamma$ , then  $A_i \cap A_j = \emptyset$ ;
- $\Gamma$  contains at most one formula of the form  $\psi$  such that  $\llbracket A \rrbracket \circ \psi \in \Delta$ , since the number  $\text{co}(\sigma)$  is uniquely determined for every  $\sigma \in D(\Delta)$ ;
- if  $\{\langle\langle A_i \rangle\rangle \circ \varphi_i, \llbracket A' \rrbracket \circ \psi\} \subseteq \Delta$  and  $\{\varphi_i, \psi\} \subseteq \Gamma$ , then  $A_i \subseteq A'$ .

Here is some intuition on the rule **(Next)** (see also [9]). This rule must ensure that for each  $\langle\langle A \rangle\rangle \circ \varphi$  from  $\mathbb{L}$  there is a respective  $A$ -action at  $\Delta$  that guarantees  $\varphi$  in the label of every successor and that for every  $\llbracket A' \rrbracket \circ \psi$  from  $\mathbb{L}$  there is a  $A'$ -co-action at  $\Delta$  that ensures  $\psi$  in the label of the respective successors.

Now, the actions at  $\Delta$  are defined so that every agent's action represents a choice of that agent of a formula from  $\mathbb{L}$  for the satisfaction of which the agent chooses to act. When all agents in some  $A_p$  choose action  $p$ , then they act together for satisfying  $\langle\langle A_p \rangle\rangle \circ \varphi_p$ , so this is the required  $A_p$ -action. As for



the co-actions, the idea is that for any fixed  $\llbracket A'_q \rrbracket \odot \psi_q$  in  $\mathbb{L}$ , all agents in the complement of  $A'_q$  may choose to act in favour of some  $\llbracket . \rrbracket$ -formula by simply selecting an action of the type  $\sigma_i \geq m$ . Then, for every  $A'_q$ -action the agents in  $\mathbb{A} - A'_q$  can synchronise their actions to ensure that the resulting action profile  $\sigma$  satisfies  $\text{co}(\sigma) = q$ , thereby ensuring  $\psi_q$  in the successor state. In fact, any agent who chooses to act *co-strategically*, i.e., in favour of a  $\llbracket . \rrbracket$ -formula, can always synchronise her action with all other agents acting co-strategically to ensure that the resulting action profile  $\sigma$  satisfies  $\text{co}(\sigma) = j$ , for any value  $j = 0, \dots, l-1$ . So, every such agent  $i$  is able, once all other agents have chosen their actions, to unilaterally enforce in the successor state any  $\psi_q$  such that  $\llbracket A'_q \rrbracket \odot \psi_q$  in  $\mathbb{L}$  and  $i \notin A'_q$ .

The rules **(SR)** and **(Next)** are applied alternatively until saturation, which is bound to occur because every label is a subset of  $\text{cl}(\eta)$ . Then the construction phase is over. The graph built in this phase is called *pretableau* for the input formula  $\eta$  and denoted by  $\mathcal{P}^\eta$ . Given a pretableau, if  $\Gamma$  is a prestate, we denote by  $\mathbf{states}(\Gamma)$  the finite set of states that are targets of  $\implies$  edges outgoing from  $\Gamma$  and if  $\Delta$  is a state we denote by  $\mathbf{prestates}(\Delta)$  the finite set of prestates that are targets of  $\longrightarrow$  edges outgoing from  $\Delta$ .

Before providing an example of how rule **(Next)** works, we give an example for the computation of the function **co**.

**Example 4.2.** Let the input formula, containing two agents, 1 and 2, be such that at some step of the pretableau construction, there is a state containing the next four primitive formulae:  $\{\langle\langle 1 \rangle\rangle \odot \varphi_1, \langle\langle 1, 2 \rangle\rangle \odot \varphi_2, \llbracket 2 \rrbracket \odot \varphi_3, \llbracket 1 \rrbracket \odot \varphi_4\}$ . The computation of the functions  $N$  and **co** and the successor prestate  $\Gamma_\sigma$  for each action profile gives:

$\sigma$	$N(\sigma)$	$\text{co}(\sigma)$	$\Gamma_\sigma$	$\sigma$	$N(\sigma)$	$\text{co}(\sigma)$	$\Gamma_\sigma$
0,0	$\emptyset$	0	$\{\varphi_1\}$	2,0	$\{1\}$	0	$\{\varphi_3\}$
0,1	$\emptyset$	0	$\{\varphi_1\}$	2,1	$\{1\}$	0	$\{\top\}$
0,2	$\{2\}$	0	$\{\varphi_1\}$	2,2	$\{1, 2\}$	0	$\{\varphi_3\}$
0,3	$\{2\}$	1	$\{\varphi_1, \varphi_4\}$	2,3	$\{1, 2\}$	1	$\{\varphi_4\}$
1,0	$\emptyset$	0	$\{\top\}$	3,0	$\{1\}$	1	$\{\top\}$
1,1	$\emptyset$	0	$\{\varphi_2\}$	3,1	$\{1\}$	1	$\{\top\}$
1,2	$\{2\}$	0	$\{\top\}$	3,2	$\{1, 2\}$	1	$\{\varphi_4\}$
1,3	$\{2\}$	1	$\{\varphi_4\}$	3,3	$\{1, 2\}$	0	$\{\varphi_3\}$

**Example 4.3.** (*Continuation of Example 4.1 for  $\theta$* ) For  $\Delta_1$ , the list of successor formulae is

$$\mathbb{L} = \langle\langle 1 \rangle\rangle \odot \langle\langle 1 \rangle\rangle p \mathcal{U} q, \langle\langle 2 \rangle\rangle \odot \langle\langle 2 \rangle\rangle (\Diamond p \wedge \Box \neg q)$$

So  $m = 2, l = 0$  and  $r_{\Delta_1} = 2$ .

As there are no unavoidable successor formulae, we do not need to compute  $N(\sigma)$  and  $\text{co}(\sigma)$ . Then,

$$\begin{aligned}\Gamma_{(0,0)} &= \{\langle\langle 1 \rangle\rangle p\mathcal{U}q\} = \Gamma_1 & \Gamma_{(1,0)} &= \{\top\} = \Gamma_3 \\ \Gamma_{(0,1)} &= \{\langle\langle 1 \rangle\rangle p\mathcal{U}q, \langle\langle 2 \rangle\rangle (\Diamond p \wedge \Box \neg q)\} = \Gamma_2 & \Gamma_{(1,1)} &= \{\langle\langle 2 \rangle\rangle (\Diamond p \wedge \Box \neg q)\} = \Gamma_4.\end{aligned}$$

For  $\Delta_2$ , the list of successor formulae is

$$\mathbb{L} = \langle\langle 1 \rangle\rangle \bigcirc \langle\langle 1 \rangle\rangle p\mathcal{U}q, \langle\langle 2 \rangle\rangle \bigcirc \langle\langle 2 \rangle\rangle \Box \neg q$$

So  $m = 2, l = 0$  and  $r_{\Delta_2} = 2$ .

Here again, we do not compute  $N(\sigma)$  and  $\text{co}(\sigma)$ . Then

$$\begin{aligned}\Gamma_{(0,0)} &= \{\langle\langle 1 \rangle\rangle p\mathcal{U}q\} = \Gamma_1 & \Gamma_{(1,0)} &= \{\top\} = \Gamma_3 \\ \Gamma_{(0,1)} &= \{\langle\langle 1 \rangle\rangle p\mathcal{U}q, \langle\langle 2 \rangle\rangle \Box \neg q\} = \Gamma_5 & \Gamma_{(1,1)} &= \{\langle\langle 2 \rangle\rangle \Box \neg q\} = \Gamma_6.\end{aligned}$$

Applying rule **(SR)** to the so-obtained prestates, we have:

$$\begin{aligned}\text{states}(\Gamma_1) &= \{\Delta_3 : \{\langle\langle 1 \rangle\rangle p\mathcal{U}q, p, \langle\langle 1 \rangle\rangle \bigcirc \langle\langle 1 \rangle\rangle p\mathcal{U}q\}, \Delta_4 : \{\langle\langle 1 \rangle\rangle p\mathcal{U}q, q, \langle\langle 1, 2 \rangle\rangle \bigcirc \top\}\}, \\ \text{states}(\Gamma_2) &= \{\Delta_5 : \{\langle\langle 1 \rangle\rangle p\mathcal{U}q, \langle\langle 2 \rangle\rangle (\Diamond p \wedge \Box \neg q), p, \neg q, \langle\langle 1 \rangle\rangle \bigcirc \langle\langle 1 \rangle\rangle p\mathcal{U}q, \langle\langle 2 \rangle\rangle \bigcirc \langle\langle 2 \rangle\rangle (\Diamond p \wedge \Box \neg q)\}, \Delta_6 : \{\langle\langle 1 \rangle\rangle p\mathcal{U}q, \langle\langle 2 \rangle\rangle (\Diamond p \wedge \Box \neg q), p, p \wedge \neg q, \neg q, \langle\langle 1 \rangle\rangle \bigcirc \langle\langle 1 \rangle\rangle p\mathcal{U}q, \langle\langle 2 \rangle\rangle \bigcirc \langle\langle 2 \rangle\rangle \Box \neg q\}\}, \\ \text{states}(\Gamma_3) &= \{\Delta_7 : \{\top, \langle\langle 1, 2 \rangle\rangle \bigcirc \top\}\}, \\ \text{states}(\Gamma_4) &= \{\Delta_8 : \{\langle\langle 2 \rangle\rangle (\Diamond p \wedge \Box \neg q), \neg q, \langle\langle 2 \rangle\rangle \bigcirc \langle\langle 2 \rangle\rangle (\Diamond p \wedge \Box \neg q)\}, \Delta_9 : \{\langle\langle 2 \rangle\rangle (\Diamond p \wedge \Box \neg q), p \wedge \neg q, \neg q, \langle\langle 2 \rangle\rangle \bigcirc \langle\langle 2 \rangle\rangle \Box \neg q\}\}, \\ \text{states}(\Gamma_5) &= \{\Delta_{10} : \{\langle\langle 1 \rangle\rangle p\mathcal{U}q, \langle\langle 2 \rangle\rangle \Box \neg q, p, \neg q, \langle\langle 1 \rangle\rangle \bigcirc \langle\langle 1 \rangle\rangle p\mathcal{U}q, \langle\langle 2 \rangle\rangle \bigcirc \langle\langle 2 \rangle\rangle \Box \neg q\}\}, \\ \text{states}(\Gamma_6) &= \{\Delta_{11} : \{\langle\langle 2 \rangle\rangle \Box \neg q, \neg q, \langle\langle 2 \rangle\rangle \bigcirc \langle\langle 2 \rangle\rangle \Box \neg q\}\}.\end{aligned}$$

The pretableau for  $\theta$  is given in Figure 5.

**Example 4.4.** (*Continuation of Example 4.1 for  $\vartheta$* ) For  $\Delta_1$ , the list of successor formulae is

$$\mathbb{L} = \langle\langle 1 \rangle\rangle \bigcirc \langle\langle 1 \rangle\rangle p\mathcal{U}q, \llbracket 2 \rrbracket \bigcirc \llbracket 2 \rrbracket (\Diamond p \wedge \Box \neg q)$$

So  $m = 1, l = 1$  and  $r_{\Delta_1} = 2$ . Therefore,

$$\begin{aligned}N(0, 0) &= \emptyset & N(1, 0) &= \{1\} \\ N(0, 1) &= \{2\} & N(1, 1) &= \{1, 2\}\end{aligned}$$

and also  $\text{co}(0, 0) = \text{co}(0, 1) = \text{co}(1, 0) = \text{co}(1, 1) = 0$ . Then,

$$\Gamma_{(0,0)} = \Gamma_{(0,1)} = \{\langle\langle 1 \rangle\rangle p\mathcal{U}q\} = \Gamma_1, \text{ and } \Gamma_{(1,0)} = \Gamma_{(1,1)} = \{\llbracket 2 \rrbracket (\Diamond p \wedge \Box \neg q)\} = \Gamma_2.$$

For  $\Delta_2$ , the list of successor formulae is

$$\mathbb{L} = \langle\langle 1 \rangle\rangle \bigcirc \langle\langle 1 \rangle\rangle p\mathcal{U}q, \llbracket 2 \rrbracket \bigcirc \llbracket 2 \rrbracket \Box \neg q$$

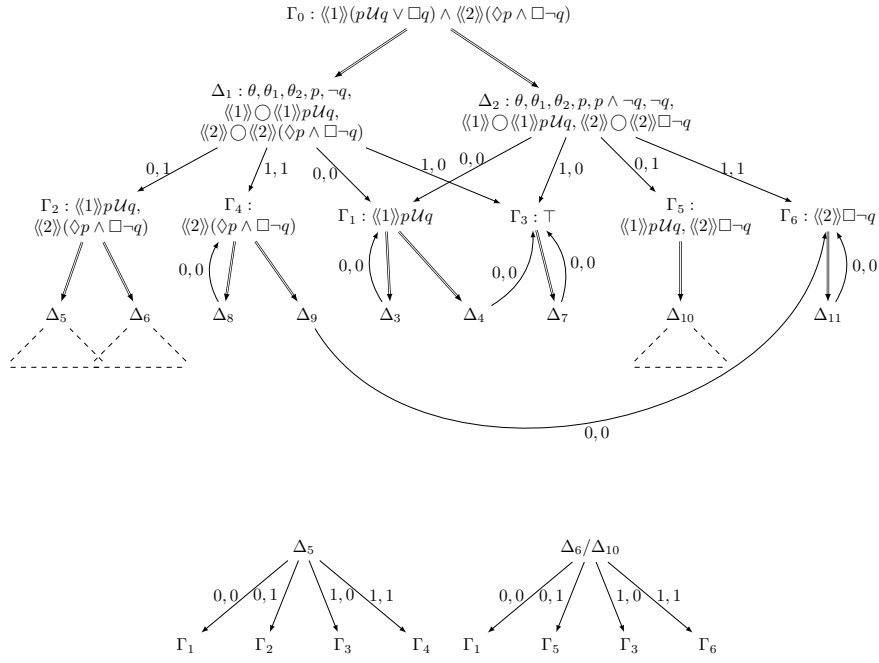


Figure 5: The pretableau for  $\theta$

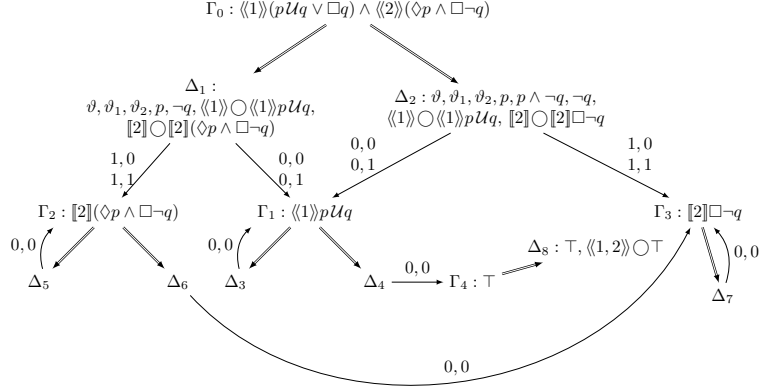


Figure 6: The pretableau for  $\vartheta$

So  $m = 1$ ,  $l = 1$  and  $r_{\Delta_2} = 2$ . Here also

$$\begin{aligned} N(0, 0) &= \emptyset & N(1, 0) &= \{1\} \\ N(0, 1) &= \{2\} & N(1, 1) &= \{1, 2\} \end{aligned}$$

and  $\text{co}(0, 0) = \text{co}(0, 1) = \text{co}(1, 0) = \text{co}(1, 1) = 0$ . Then,

$$\Gamma_{(0,0)} = \Gamma_{(0,1)} = \{\langle\langle 1 \rangle\rangle p\mathcal{U}q\} = \Gamma_1, \text{ and } \Gamma_{(1,0)} = \Gamma_{(1,1)} = \{\llbracket 2 \rrbracket \Box \neg q\} = \Gamma_3.$$

In the same way, we obtain:

$$\begin{aligned} \mathbf{states}(\Gamma_1) &= \{\Delta_3 : \{\langle\langle 1 \rangle\rangle p\mathcal{U}q, p, \langle\langle 1 \rangle\rangle \Box \langle\langle 1 \rangle\rangle p\mathcal{U}q\}, \Delta_4 : \{\langle\langle 1 \rangle\rangle p\mathcal{U}q, q, \langle\langle 1, 2 \rangle\rangle \Box \top\}\}; \\ \mathbf{states}(\Gamma_2) &= \{\Delta_5 : \{\llbracket 2 \rrbracket (\Diamond p \wedge \Box \neg q), \neg q, \llbracket 2 \rrbracket \Box \llbracket 2 \rrbracket (\Diamond p \wedge \Box \neg q)\}, \Delta_6 : \{\llbracket 2 \rrbracket (\Diamond p \wedge \Box \neg q), p \wedge \neg q, p, \neg q, \llbracket 2 \rrbracket \Box \llbracket 2 \rrbracket \Box \neg q\}\}; \\ \mathbf{states}(\Gamma_3) &= \{\Delta_7 : \{\llbracket 2 \rrbracket \Box \neg q, \neg q, \llbracket 2 \rrbracket \Box \llbracket 2 \rrbracket \Box \neg q\}\}; \\ \mathbf{states}(\Gamma_4) &= \{\Delta_8 : \{\top, \langle\langle 1, 2 \rangle\rangle \Box \top\}\}. \end{aligned}$$

The pretableau for  $\vartheta$  is given in Figure 6.

## 4.2 The prestate and state elimination phases. Eventualities

First, we remove from  $\mathcal{P}^\eta$  all the prestates and the  $\Rightarrow$  edges, as follows. For every prestate  $\Gamma$  in  $\mathcal{P}^\eta$  put  $\Delta \xrightarrow{\sigma} \Delta'$  for all states  $\Delta$  in  $\mathcal{P}^\eta$  with  $\Delta \xrightarrow{\sigma} \Gamma$  and all  $\Delta' \in \mathbf{states}(\Gamma)$ ; then, remove  $\Gamma$  from  $\mathcal{P}^\eta$ . The graph obtained after eliminating all prestates is called the *initial tableau*, denoted by  $\mathcal{T}_0^\eta$ . The initial tableau for the formula  $\theta$  in our running example is given on Figure 7 and the initial tableau for  $\vartheta$  is given on Figure 8.

The elimination phase starts with  $\mathcal{T}_0^\eta$  and goes through stages. At stage  $n+1$  we remove exactly one state from the tableau  $\mathcal{T}_n^\eta$  obtained at the previous

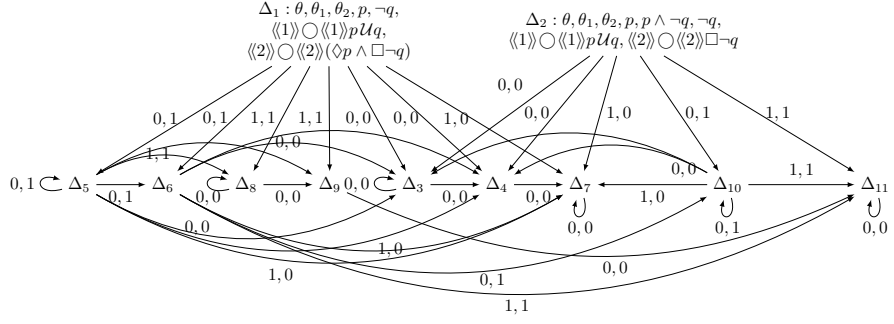


Figure 7: The initial tableau for  $\theta$

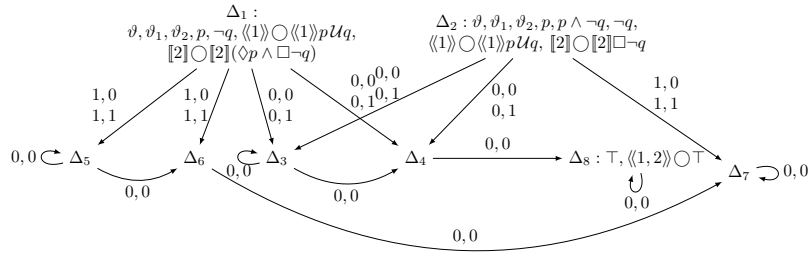


Figure 8: The initial and final tableau for  $\vartheta$

stage, by applying one of the elimination rules described below, thus obtaining the tableau  $\mathcal{T}_{n+1}^\eta$ . The set of states of  $\mathcal{T}_m^\eta$  is noted  $S_m^\eta$ .

The first elimination rule (**ER1**), defined below, is used to eliminate all states with missing successors for some action vectors determined by the rule (**Next**). If, due to a previous state elimination, any state has an outgoing action vector for which the corresponding successor state is missing, we delete the state. The reason is clear: if  $\Delta$  is to be satisfiable, then for each  $\sigma \in D(\Delta)$  there should exist a satisfiable  $\Delta'$  that  $\Delta$  reaches via  $\sigma$ . Formally, the rule is stated as follows, where  $D(\Delta)$  is defined in the rule (**Next**):

**Rule (ER1):** If, for some  $\sigma \in D(\Delta)$ , all states  $\Delta'$  with  $\Delta \xrightarrow{\sigma} \Delta'$  have been eliminated at earlier stages, then obtain  $\mathcal{T}_{n+1}^\eta$  by eliminating  $\Delta$  (together with its adjacent edges) from  $\mathcal{T}_n^\eta$ .

The aim of the next elimination rule is to make sure that there are no *unrealized eventualities*. In ATL there are only two kinds of eventualities :  $\langle\langle A \rangle\rangle\varphi\mathcal{U}\psi$  and  $\llbracket A \rrbracket\varphi\mathcal{U}\psi$ . The situation is more complex in  $\text{ATL}^+$ . For instance, should the formula  $\langle\langle A \rangle\rangle(\Box\varphi \vee \psi_1\mathcal{U}\psi_2)$  be considered an eventuality? Our solution for  $\text{ATL}^+$  is to consider all  $\gamma$ -formulae as *potential eventualities*. In order to properly define the notion of *realization of a potential eventuality* we first define a Boolean-valued function *Real* that takes as arguments two elements: an  $\text{ATL}^+$  path-formula  $\Phi$  and a set  $\Theta$  of  $\text{ATL}^+$  state-formulae. This function allows us to check the realization of a potential eventuality of the form  $\langle\langle A \rangle\rangle\Phi$  and  $\llbracket A \rrbracket\Phi$  (where  $\Phi$  is the first argument of *Real*) at a given state labelled by  $\Theta$  (where  $\Theta$  is the second argument of *Real*).

- $Real(\Phi \wedge \Psi, \Theta) = Real(\Phi, \Theta) \wedge Real(\Psi, \Theta)$
- $Real(\Phi \vee \Psi, \Theta) = Real(\Phi, \Theta) \vee Real(\Psi, \Theta)$
- $Real(\varphi, \Theta) = \text{true}$  iff  $\varphi \in \Theta$
- $Real(\Box\varphi, \Theta) = \text{false}$
- $Real(\Box\varphi, \Theta) = \text{true}$  iff  $\varphi \in \Theta$
- $Real(\varphi\mathcal{U}\psi, \Theta) = \text{true}$  iff  $\psi \in \Theta$

**Definition 4.1** (Descendant potential eventualities). Let  $\xi \in \Delta$  be a potential eventuality of the form  $\langle\langle A \rangle\rangle\Phi$  or  $\llbracket A \rrbracket\Phi$ . Suppose the  $\gamma$ -component  $\gamma(\xi, \Delta)$  in  $\Delta$  linked to  $\xi$  is, respectively, of the form  $\psi \wedge \langle\langle A \rangle\rangle\Box\langle\langle A \rangle\rangle\Psi$  or  $\psi \wedge \llbracket A \rrbracket\Box\llbracket A \rrbracket\Psi$ . Then the *successor potential eventuality* of  $\xi$  w.r.t.  $\gamma(\xi, \Delta)$  is the  $\gamma$ -formula  $\langle\langle A \rangle\rangle\Psi$  (resp.  $\llbracket A \rrbracket\Psi$ ) and it will be denoted by  $\xi_\Delta^1$ . The notion of *descendant potential eventuality of  $\xi$  of degree  $d$* , for  $d > 1$ , is defined inductively as follows:

- any successor eventuality of  $\xi$  (w.r.t. some  $\gamma$ -component of  $\xi$ ) is a descendant eventuality of  $\xi$  of degree 1;
- any successor eventuality of a descendant eventuality  $\xi^n$  of  $\xi$  of degree  $n$  is a descendant eventuality of  $\xi$  of degree  $n + 1$ .

We will also consider  $\xi$  to be a descendant eventuality of itself of degree 0.

**Example 4.5.** Let  $\xi = \langle\langle A \rangle\rangle(\Box p \vee q\mathcal{U}r)$  be a potential eventuality such that  $\xi \in \Theta$ , where  $\Theta$  is the labelling of a state  $\Delta$ . Let us see the different cases that can occur and the corresponding result of the function *Real*.

1.  $p \in \Theta$  and  $r \in \Theta$ . In that case,  $\text{Real}(\Box p, \Theta) = \text{true}$  and  $\text{Real}(q\mathcal{U}r, \Theta) = \text{true}$ , so  $\text{Real}(\Box p \vee q\mathcal{U}r, \Theta) = \text{true}$ . This is indeed correct since  $q\mathcal{U}r$  is immediately realized.
2.  $p \notin \Theta$  and  $r \in \Theta$ . This case is similar to the previous one even if  $\text{Real}(\Box p, \Theta) = \text{false}$ , indeed  $\text{Real}(\Box p \vee q\mathcal{U}r, \Theta) = \text{true}$ .
3.  $p \notin \Theta$  and  $r \notin \Theta$ . Here  $\text{Real}(\Box p \vee q\mathcal{U}r, \Theta) = \text{false}$  and the potential eventuality  $\xi$  is not immediately realized. The rule **(SR)** guarantees that  $q \in \Theta$ . This case means that the part  $\Box p$  of  $\xi$  is skipped and the part  $q\mathcal{U}r$  will be continued. Therefore the next potential eventuality to be realized is  $\langle\langle A \rangle\rangle q\mathcal{U}r$ . The immediate realization of this new potential eventuality will be checked again at next states.
4.  $p \in \Theta$  and  $r \notin \Theta$ . The potential eventuality  $\xi$  is immediately realized since  $\text{Real}(\Box p, \Theta) = \text{true}$ , but two sub-cases can be distinguished to explain why this is correct:
  - (a)  $q \in \Theta$ . Here both possibilities to do either  $\Box p$  or  $q\mathcal{U}r$  are kept. So the successor potential eventuality is  $\langle\langle A \rangle\rangle(\Box p \vee q\mathcal{U}r)$  and its immediate realization will be checked again at next states.
  - (b)  $q \notin \Theta$ . This means that only the part  $\Box p$  will be kept and the successor potential eventuality is  $\langle\langle A \rangle\rangle\Box p$ . This case can be declared immediately realized since the construction rules of the tableau guarantees that  $\langle\langle A \rangle\rangle\Box p$  is correctly treated.

**Example 4.6.** (*Continuation of Example 4.3*) In  $\Delta_5$  we have  $\xi = \langle\langle 1 \rangle\rangle(p\mathcal{U}q \vee \Box q)$  with  $\text{Real}(p\mathcal{U}q \vee \Box q, \Delta_1) = \text{Real}(p\mathcal{U}q, \Delta_1) \vee \text{Real}(\Box q, \Delta_1) = \text{false} \vee \text{false} = \text{false}$ , since  $q \notin \Delta_1$ , and  $\xi' = \langle\langle 2 \rangle\rangle(\Diamond p \wedge \Box \neg q)$  with  $\text{Real}(\Diamond p \wedge \Box \neg q, \Delta_1) = \text{Real}(\Diamond p, \Delta_1) \wedge \text{Real}(\Box \neg q, \Delta_1) = \text{true} \wedge \text{true} = \text{true}$  since  $p, \neg q \in \Delta_1$ .

The successor eventuality of  $\xi = \langle\langle 1 \rangle\rangle(p\mathcal{U}q \vee \Box q)$  w.r.t  $\gamma(\xi, \Delta_1)$  is  $\xi_{\Delta_1}^1 = \langle\langle 1 \rangle\rangle p\mathcal{U}q$  in  $\Delta_3, \Delta_4, \Delta_5, \Delta_6$ . For each  $n > 1$ , the descendant eventuality of degree  $n$  of  $\xi$  w.r.t  $\gamma(\xi, \Delta_1)$  is  $\xi_{\Delta_1}^n = \xi_{\Delta_1}^1$  in  $\Delta_3, \Delta_4, \Delta_5, \Delta_6, \Delta_{10}$ . The successor eventuality of  $\xi' = \langle\langle 2 \rangle\rangle(\Diamond p \wedge \Box \neg q)$  w.r.t  $\gamma(\xi', \Delta_5)$  is  $\xi_{\Delta_5}^1 = \langle\langle 2 \rangle\rangle(\Diamond p \wedge \Box \neg q)$  in  $\Delta_5, \Delta_6, \Delta_8$ . For each  $n > 1$ , the descendant eventualities of degree  $n$  of  $\xi'$  w.r.t  $\gamma(\xi', \Delta_5)$  are  $\xi_{\Delta_5}^n = \xi_{\Delta_5}^1$  in  $\Delta_5, \Delta_6, \Delta_8$  and  $\Delta_9$ ; and  $\xi_{\Delta_5}^n = \langle\langle 2 \rangle\rangle \Box \neg q$  in  $\Delta_{10}$  and  $\Delta_{11}$ .

Now, let  $\mathbb{L} = \langle\langle A_0 \rangle\rangle \circ \varphi_0, \dots, \langle\langle A_{m-1} \rangle\rangle \circ \varphi_{m-1}, \llbracket A'_0 \rrbracket \circ \psi_0, \dots, \llbracket A'_{l-1} \rrbracket \circ \psi_{l-1}$  be the list of all primitive successor formulae of  $\Delta \in S_0^\eta$ , induced as part of an application of **(Next)**. We will use the following notation:

$$\begin{aligned} D(\Delta, \langle\langle A_p \rangle\rangle \circ \varphi_p) &:= \{\sigma \in D(\Delta) \mid \sigma_a = p \text{ for every } a \in A_p\} \\ D(\Delta, \llbracket A'_q \rrbracket \circ \psi_q) &:= \{\sigma \in D(\Delta) \mid \text{co}(\sigma) = q \text{ and } \mathbb{A} - A'_q \subseteq N(\sigma)\} \end{aligned}$$

Next, we will define recursively what it means for an eventuality  $\xi$  to be realized at a state  $\Delta$  of a tableau  $\mathcal{T}_n^\eta$ , followed by our second elimination rule.

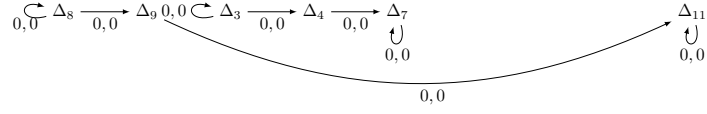


Figure 9: The final tableau for  $\theta$

**Definition 4.2** (Realization of potential eventualities). Let  $\Delta \in S_n^\eta$  and  $\xi \in \Delta$  be a potential eventuality of the form  $\langle\langle A \rangle\rangle\Phi$  or  $\llbracket A \rrbracket\Phi$ . Then:

1. If  $\text{Real}(\Phi, \Delta) = \text{true}$  then  $\xi$  is realized at  $\Delta$  in  $\mathcal{T}_n^\eta$ .
2. Else, let  $\xi_\Delta^1$  be the successor potential eventuality of  $\xi$  w.r.t.  $\gamma(\xi, \Delta)$ . If for every  $\sigma \in D(\Delta, \langle\langle A \rangle\rangle \circ \xi_\Delta^1)$  (resp.  $\sigma \in D(\Delta, \llbracket A \rrbracket \circ \xi_\Delta^1)$ ), there exists  $\Delta' \in \mathcal{T}_n^\eta$  with  $\Delta \xrightarrow{\sigma} \Delta'$  and  $\xi_\Delta^1$  is realized at  $\Delta'$  in  $\mathcal{T}_n^\eta$ , then  $\xi$  is realized at  $\Delta$  in  $\mathcal{T}_n^\eta$ .

**Rule (ER2):** If  $\Delta \in S_n^\eta$  contains a potential eventuality that is not realized at  $\Delta \in \mathcal{T}_n^\eta$ , then obtain  $\mathcal{T}_{n+1}^\eta$  by removing  $\Delta$  (together with its adjacent edges) from  $S_n^\eta$ .

**Example 4.7.** (*Continuation of Example 4.6*) The potential eventuality  $\xi'' = \langle\langle 1 \rangle\rangle(p\mathcal{U}q)$  is not realized at  $\Delta_5$ , so by Rule (ER2) we remove the state  $\Delta_5$  from  $\mathcal{T}_0^\theta$  and obtain the tableau  $\mathcal{T}_1^\theta$ . The same applies to  $\Delta_6$  for  $\xi''$ , so we also remove  $\Delta_6$  from  $\mathcal{T}_1^\theta$  and obtain  $\mathcal{T}_2^\theta$  with Rule (ER2). In  $\mathcal{T}_2^\theta$  there is no more move vector  $(0, 1)$  for the state  $\Delta_1$ , so by Rule (ER1) we remove  $\Delta_1$  from  $\mathcal{T}_2^\theta$  and obtain  $\mathcal{T}_3^\theta$ . In the same way,  $\Delta_{10}$  is removed by Rule (ER2) and  $\Delta_2$  by Rule (ER1).

For the case of  $\vartheta$ , it is easy to see that no state gets eliminated, so the final tableau is the same as the initial one.

The elimination phase is completed when no more applications of elimination rules are possible. Then we obtain the *final tableau* for  $\eta$ , denoted by  $\mathcal{T}^\eta$ . It is declared *open* if  $\eta$  belongs to some state in it, otherwise *closed*. The procedure for deciding satisfiability of  $\eta$  returns “No” if  $\mathcal{T}^\eta$  is closed, “Yes” otherwise.

**Example 4.8.** (*Continuation of Example 4.7*) At the end of the elimination phase,  $\Delta_1$  and  $\Delta_2$  are no longer in  $\mathcal{T}^\theta$ . Thus  $\mathcal{T}^\theta$  is closed and the formula  $\theta = \langle\langle 1 \rangle\rangle(p\mathcal{U}q \vee \Box q) \wedge \langle\langle 2 \rangle\rangle(\Diamond p \wedge \Box \neg q)$  is declared unsatisfiable. The final tableau for  $\theta$  is given on Figure 9.

Respectively, the final tableau for  $\vartheta$  is open, hence  $\vartheta$  is declared satisfiable. Indeed, a CGM can be extracted from the final tableau. We will explain in Section 6.2 how this can be done in a systematic way.

## 5 Termination and soundness

The termination of the tableau procedure is straightforward, as there are only finitely many states and prestates that can be added in the construction phase.



**Theorem 5.1.** *The tableau method for  $\text{ATL}^+$  is sound.*

Soundness of the tableau procedure with respect to unsatisfiability means that if a formula is satisfiable then its final tableau is open. To prove that, we essentially follow the same procedure as in the soundness proof for the tableau-based decision procedure for  $\text{ATL}$  in [9].

The soundness proof consists of three main claims. First, we show that when a prestate  $\Gamma$  is satisfiable then at least one of the states in  $\mathbf{states}(\Gamma)$  is satisfiable. Then, we prove that when a state  $\Delta$  is satisfiable then all the prestates in  $\mathbf{prestates}(\Delta)$  are satisfiable. Finally, we show that no satisfiable states are eliminated in the elimination phase. Below, we take the input formula of the tableau procedure to be  $\eta$ .

The first step of the proof consists in showing that rule **(SR)** is sound:

**Lemma 5.1.** *Let  $\Gamma$  be a prestate of  $\mathcal{P}^\eta$  and let  $\mathcal{M}, s \models \Gamma$  for some CGM  $\mathcal{M}$  and some  $s \in \mathcal{M}$ . Then,  $\mathcal{M}, s \models \Delta$  holds for at least one  $\Delta \in \mathbf{states}(\Gamma)$ .*

*Proof.* Straightforward from Proposition 3.1. □ □

The aim of the next two lemmas is to show that the rule **(Next)** creates only satisfiable prestates from satisfiable states.

We recall that we use  $\mathbf{act}_A(s)$  to denote the set of all  $A$ -actions that can be played by the coalition  $A$  at state  $s$ , i.e.  $\mathbf{act}_A(s) = \Pi_{a \in A} \mathbf{act}_a(s)$ . We also use  $\mathbf{act}_A^c(s)$  to denote the set of all  $A$ -co-actions available at state  $s$  and  $\sigma_A^c$  for an element of this set. Let  $\sigma_A \in \mathbf{act}_A(s)$ . We say that an action profile  $\sigma_\mathbb{A}$  extends an  $A$ -action  $\sigma_A$ , denoted by  $\sigma_\mathbb{A} \supseteq \sigma_A$ , if  $\sigma_\mathbb{A}(a) = \sigma_A(a)$  for every  $a \in A$ . We also use  $\mathbf{Out}(s, \sigma_A)$  to denote the set of all states  $s'$  for which there exists an action profile  $\sigma_\mathbb{A} \in \mathbf{act}_\mathbb{A}(s)$  that extends  $\sigma_A$  and such that  $\mathbf{out}(s, \sigma_\mathbb{A}) = s'$ . We define in a same way  $\supseteq$  and  $\mathbf{Out}(s, \sigma_A^c)$  for an  $A$ -co-action  $\sigma_A^c \in \mathbf{act}_A^c(s)$ .

The following lemma states a semantic property, independent of the tableau construction.

**Lemma 5.2.** *Let  $\Theta = \{\langle\langle A_1 \rangle\rangle \bigcirc \varphi_1, \dots, \langle\langle A_m \rangle\rangle \bigcirc \varphi_m, \llbracket A' \rrbracket \bigcirc \psi\}$  be a set of formulae such that  $A_i \cap A_j = \emptyset$  for every  $1 \leq i, j \leq m$ ,  $i \neq j$  and  $A_i \subseteq A'$  for every  $1 \leq i \leq m$ . Let  $\mathcal{M}, s \models \Theta$  for some GCM  $\mathcal{M}$  and  $s \in \mathcal{M}$ . Let  $\sigma_{A_i} \in \mathbf{act}_{A_i}(s)$  be an  $A_i$ -action witnessing the truth of  $\langle\langle A_i \rangle\rangle \bigcirc \varphi_i$  at  $s$ , for each  $1 \leq i \leq m$ , and let, finally,  $\sigma_{A'}^c \in \mathbf{act}_{A'}^c(s)$  be an  $A'$ -co-action witnessing the truth of  $\llbracket A' \rrbracket \bigcirc \psi$  at  $s$ . Then there exists  $s' \in \mathbf{Out}(s, \sigma_{A_1}) \cap \dots \cap \mathbf{Out}(s, \sigma_{A_m}) \cap \mathbf{Out}(s, \sigma_{A'}^c)$  such that  $\mathcal{M}, s' \models \{\varphi_1, \dots, \varphi_m, \psi\}$ .*

*Proof.* Let  $A = A_1 \cup \dots \cup A_m$ . Since  $A_i \cap A_j = \emptyset$  for every  $1 \leq i, j \leq m$ , the actions  $\sigma_{A_1}, \dots, \sigma_{A_m}$  can be combined to get an  $A$ -action  $\sigma_A$ . This last can be arbitrarily extended to an  $A'$ -action  $\sigma_{A'}$  because  $A_i \subseteq A'$  for every  $1 \leq i \leq m$ . Finally, the so obtained  $\sigma_{A'}$  can be completed by the  $A'$ -co-action  $\sigma_{A'}^c$ . The resulting action  $\sigma_\mathbb{A}$  leads from  $s$  to the desired  $s'$ . □ □

The next lemma states that satisfiability propagates from states to their successor prestates created via rule **(Next)**.

**Lemma 5.3.** *If  $\Delta \in \mathcal{P}^\eta$  is a satisfiable state then all the prestates  $\Gamma$  obtained by applying the rule **(Next)** are satisfiable.*

*Proof.* Follows by induction on the number of steps in the construction of the tableau, and from Lemma 5.2 and Remark 4.1.  $\square$   $\square$

Thus, the rule **(SR)** generates at least one satisfiable state from a satisfiable prestate and that the rule **(Next)** generates only satisfiable prestates from a satisfiable state. Hence, we can conclude that the construction phase of the tableau procedure is sound.

We now move to the elimination phase.

**Lemma 5.4.** *Let  $\Delta$  be a state in  $\mathcal{T}_n^\eta$ . If  $\Delta$  is satisfiable then Rule **(ER1)** cannot eliminate  $\Delta$  from  $\mathcal{T}_n^\eta$ .*

*Proof.* By Lemma 5.2 a satisfiable state  $\Delta$  generates only satisfiable successor prestates, and, by Lemma 5.3, each of these prestates in turn generates at least one satisfiable state. Therefore, if  $\Delta$  is satisfiable then for every action profile  $\sigma_{\mathbb{A}} \in \text{act}_{\mathbb{A}}(s)$  there is a state  $\Delta'$  such that  $\Delta \xrightarrow{\sigma_{\mathbb{A}}} \Delta'$ . Therefore the Rule **(ER1)** cannot eliminate a satisfiable state.  $\square$   $\square$

It remains to be proved that a satisfiable state cannot be eliminated by Rule **(ER2)**, either. We recall that Rule **(ER2)** eliminates each state containing an eventuality that is not realized at that state. So we need to prove that if a state  $\Delta$  is satisfiable, then every eventuality  $\xi \in \Delta$  is realized at  $\Delta$  throughout the elimination phase.

Note that the structure underlying a tableau can be seen as a non-deterministic CGS, where edges outgoing from a tableau state can lead to different successors even if they are labelled by the same action vector. The following two definitions will be used to extract deterministic transitions from non-deterministic ones.

**Definition 5.1** (Outcome set of  $\sigma_A$  at  $s$ ). Let  $\mathcal{S}$  be a non-deterministic concurrent game structure, let  $s$  be a state and let  $\sigma_A \in \text{act}_A(s)$ . An *outcome set* of  $\sigma_A$  at  $s$  is a set of states  $X$  such that for every  $\sigma_{\mathbb{A}} \sqsupseteq \sigma_A$  there exists exactly one  $s' \in X$  such that  $s \xrightarrow{\sigma_{\mathbb{A}}} s'$ .

**Definition 5.2** (Outcome set of  $\sigma_A^c$  at  $s$ ). Let  $\mathcal{S}$  be a non-deterministic concurrent game structure, let  $s$  be a state and let  $\sigma_A^c \in \text{act}_A^c(s)$ . An *outcome set* of  $\sigma_A^c$  at  $s$  is a set of states  $X$  such that for every  $\sigma_A \in \text{act}_A(s)$  there exists exactly one  $s' \in X$  such that  $s \xrightarrow{\sigma_A^c(\sigma_A)} s'$ .

In particular, both definitions above can be applied to a tableau, where the states  $s$  and  $s'$  are taken to be tableau states  $\Delta$  and  $\Delta'$ .

Some notation. Consider a concurrent game structure  $\mathcal{S}$  which is state-labelled by a set  $\Theta$  of state formulae of  $\text{ATL}^+$  and suppose that the elements of  $\Theta$  are listed by any enumeration  $E$  where enforceable next-time formulae appear before unavoidable next-time formulae, in particular, in the list  $\mathbb{L}$  given in the definition of Rule **(Next)**. Then:

1. Whenever we write  $\langle\langle A_p \rangle\rangle \circ \varphi_p \in \Theta$ , we mean that  $\langle\langle A_p \rangle\rangle \circ \varphi_p$  is the  $p$ -th enforceable next-time formula according to  $E$ . In particular, when  $\mathcal{S}$  is a tableau,  $E$  is usually assumed to be the listing of the successor formulae of  $\Theta$  induced by the application of the rule **(Next)** to  $\Theta$ .

We use the notation  $\llbracket A'_q \rrbracket \circ \psi_q \in \Theta$  likewise.

2. Given  $\langle\langle A_p \rangle\rangle \circ \varphi \in \Theta$ , we denote by  $\sigma_{A_p}[\langle\langle A_p \rangle\rangle \circ \varphi]$  a (somehow selected)  $A$ -action enforcing  $\varphi$  in any associated successor state.

In particular, when  $\mathcal{S}$  is a tableau, we denote by  $\sigma_{A_p}[\langle\langle A_p \rangle\rangle \circ \varphi_p]$  the unique  $A_p$ -action  $\sigma_{A_p} \in \text{act}_{A_p}(\Theta)$  in the tableau such that  $\sigma_{A_p}(a) = p$  for every  $a \in A_p$ .

3. Likewise, given a formula  $\llbracket A'_q \rrbracket \circ \psi \in \Theta$ , where  $A'_q \neq \mathbb{A}$ , we denote by  $\sigma_{A'_q}^c[\llbracket A'_q \rrbracket \circ \psi]$  a (somehow selected)  $A'_q$ -co-action  $\sigma_{A'_q}^c$  enforcing  $\psi$  in any associated successor state.

In particular, when  $\mathcal{S}$  is a tableau, we denote by  $\sigma_{A'_q}^c[\llbracket A'_q \rrbracket \circ \psi_q]$  the unique  $A'_q$ -co-action  $\sigma_{A'_q}^c \in \text{act}_{A'_q}^c(\Delta)$  in the tableau satisfying the following condition (with notation referring to the definition of rule **(Next)**):  $\text{co}(\sigma_{A'_q}^c(\sigma_{A'_q})) = q$  and  $\mathbb{A} - A'_q \subseteq N(\sigma_{A'_q}^c(\sigma_{A'_q}))$  for every  $\sigma_{A'_q} \in \text{act}_{A'_q}(\Delta)$ .

In order to prove that the rule **(ER2)** does not eliminate any satisfiable states, we need to show that if a tableau  $\mathcal{T}_n^\eta$  contains a state  $\Delta$  that is satisfiable and contains an eventuality  $\xi$ , then  $\xi$  is realized at  $\Delta$ . Thus we prove that  $\mathcal{T}_n^\eta$  “contains” a structure (more precisely, a tree) that “witnesses” the realization of  $\xi$  at  $\Delta$  in  $\mathcal{T}_n^\eta$ . This tree will emulate a tree of runs effected by a strategy or a co-strategy that “realizes” an eventuality in a model. This simulation is done step-by-step, and each step, i.e.  $A$ -action (in the case of  $\langle\langle A \rangle\rangle \Phi$ ) or  $A$ -co-action (in the case of  $\llbracket A \rrbracket \Phi$ ) corresponds to a tableau action or co-action associated with a respective eventuality. The fact that this step-by-step simulation can be done is proved in the next two lemmas (together with their corollaries).

**Lemma 5.5.** *Let  $\langle\langle A_p \rangle\rangle \circ \varphi_p \in \Delta \in S_n^\eta$  and let  $\mathcal{M}, s \models \Delta$  for some CGM  $\mathcal{M}$  and state  $s \in \mathcal{M}$ . Let, furthermore,  $\sigma_{A_p} \in \text{act}_{A_p}(s)$  be an  $A_p$ -action witnessing the truth of  $\langle\langle A_p \rangle\rangle \circ \varphi_p$  at  $s$ . Then, there exists in  $\mathcal{T}_n^\eta$  an outcome set  $X$  of  $\sigma_{A_p}[\langle\langle A_p \rangle\rangle \circ \varphi_p]$  such that for each  $\Delta' \in X$  there exists  $s' \in \text{Out}(s, \sigma_{A_p})$  such that  $\mathcal{M}, s' \models \Delta'$ .*

*Proof.* We consider the following set of prestates (from the pretableau construction):

$$Y = \{\Gamma \in \text{prestates}(\Delta) \mid \Delta \xrightarrow{\mathbb{A}} \Gamma \text{ for some } \sigma_{\mathbb{A}} \sqsupseteq \sigma_{A_p}[\langle\langle A_p \rangle\rangle \circ \varphi_p]\}$$

For every  $\Gamma \in Y$ , it follows immediately from the rule **(Next)** that  $\Gamma$  (which must contain  $\varphi_p$ ) is either of the form

$$\{\varphi_1, \dots, \varphi_m, \psi\}, \text{ where } \{\langle\langle A_1 \rangle\rangle \circ \varphi_1, \dots, \langle\langle A_m \rangle\rangle \circ \varphi_m, \llbracket A' \rrbracket \circ \psi\} \subseteq \Delta,$$

or of the form

$\{\varphi_1, \dots, \varphi_m\}$  where  $\{\langle\langle A_1 \rangle\rangle \circ \varphi_1, \dots, \langle\langle A_m \rangle\rangle \circ \varphi_m\} \subseteq \Delta$ .

We can reduce the latter case to the former by adding the valid formula  $\llbracket A \rrbracket \circ \top$  (equivalent to  $\langle\langle \emptyset \rangle\rangle \circ \top$ ).

Since  $\mathcal{M}, s \models \Delta$ , by Lemma 5.2, there exists  $s' \in \text{Out}(s, \sigma_{A_p})$  with  $\mathcal{M}, s' \models \Gamma$ . Then  $\Gamma$  can be extended to a fully expanded set  $\Delta'$  containing at least one successor formula ( $\langle\langle A \rangle\rangle \circ \top$ , if nothing else) such that  $\mathcal{M}, s' \models \Delta'$ . This is done by choosing, for every  $\beta$ - or  $\gamma$ -formula to be processed in the procedure computing the family of full expansions, a disjunct, resp. a  $\gamma$ -component, that is actually true in  $\mathcal{M}$  at  $s'$  (if there are several such options, the choice is arbitrary) and adding it to the current set.  $\square$   $\square$

**Corollary 1.** *Let  $\langle\langle A_p \rangle\rangle \circ \varphi_p \in \Delta$  for  $\Delta \in S_n^\eta$  and let  $\mathcal{M}, s \models \Delta$  for some CGM  $\mathcal{M}$  and state  $s \in \mathcal{M}$ . Let, furthermore,  $\sigma_{A_p} \in \text{act}_{A_p}(s)$  be an  $A_p$ -action witnessing the truth of  $\langle\langle A_p \rangle\rangle \circ \varphi_p$  at  $s$  and let  $\chi \in \text{cl}(\eta)$  be a  $\beta$ -formula (resp. a  $\gamma$ -formula) and  $\psi$  be one of its  $\beta$ -components (resp.  $\gamma$ -components). Then there exists in  $\mathcal{T}_n^\eta$  an outcome set  $X_\psi$  of  $\sigma_{A_p}[\langle\langle A_p \rangle\rangle \circ \varphi_p]$  such that for every  $\Delta' \in X_\psi$  there exists  $s' \in \text{Out}(s, \sigma_{A_p})$  such that  $\mathcal{M}, s' \models \Delta'$ , and moreover, if  $\mathcal{M}, s' \models \psi$ , then  $\psi \in \Delta'$ .*

*Proof.* Construct  $X_\psi$  just like  $X$  was constructed in the proof of the preceding lemma, with a single modification: when dealing with the formula  $\chi$ , instead of choosing arbitrarily between the different options for  $\psi$ , choose  $\psi$  which is true at  $s'$ .  $\square$   $\square$

Likewise, we obtain the following for unavoidable formulae:

**Lemma 5.6.** *Let  $\llbracket A'_q \rrbracket \circ \psi_q \in \Delta \in S_n^\eta$  and let  $\mathcal{M}, s \models \Delta$  for some CGM  $\mathcal{M}$  and state  $s \in \mathcal{M}$ . Let, furthermore,  $\sigma_{A'_q}^c \in \text{act}_{A'_q}^c(s)$  be an  $A'_q$ -co-action witnessing the truth of  $\llbracket A'_q \rrbracket \circ \psi_q$  at  $s$ . Then, there exists in  $\mathcal{T}_n^\eta$  an outcome set  $X$  of  $\sigma_{A'_q}^c[\llbracket A'_q \rrbracket \circ \psi_q]$  such that for each  $\Delta' \in X$  there exists  $s' \in \text{Out}(s, \sigma_{A'_q}^c)$  such that  $\mathcal{M}, s' \models \Delta'$ .*

The proof is analogous to the proof of Lemma 5.5.

**Corollary 2.** *Let  $\llbracket A'_q \rrbracket \circ \psi_q \in \Delta \in S_n^\eta$  and let  $\mathcal{M}, s \models \Delta$  for some CGM  $\mathcal{M}$  and state  $s \in \mathcal{M}$ . Let, furthermore,  $\sigma_{A'_q}^c \in \text{act}_{A'_q}^c(s)$  be an  $A'_q$ -co-action witnessing the truth of  $\llbracket A'_q \rrbracket \circ \psi_q$  at  $s$  and let  $\chi \in \text{cl}(\eta)$  be a  $\beta$ -formula (resp. a  $\gamma$ -formula), whose associated  $\beta_i$ -component ( $i \in \{1, 2\}$ ) (resp.  $i$ -th  $\gamma$ -component ( $i \geq 1$ )) is  $\chi_i$ . Then there exists in  $\mathcal{T}_n^\eta$  an outcome set  $X_{\chi_i}$  of  $\sigma_{A'_q}^c[\llbracket A'_q \rrbracket \circ \psi_q]$  such that for every  $\Delta' \in X_{\chi_i}$  there exists  $s' \in \text{Out}(s, \sigma_{A'_q}^c)$  such that  $\mathcal{M}, s' \models \Delta'$ , and moreover, if  $\mathcal{M}, s' \models \chi_i$ , then  $\chi_i \in \Delta'$ .*

In what follows we make use of the notion of *tree*. In our context, we use such a term as a synonym of “directed, connected, and acyclic graph, every node of which, except the root, has exactly one incoming edge”. We denote a tree as a pair  $(R, \rightarrow)$ , where  $R$  is the set of nodes and  $\rightarrow$  is the parent-child relation (the edges).

The first kind of tree that we define is the so-called *realization witness tree*. Intuitively, such tree witnesses the satisfaction of a given potential eventuality  $\xi$  at a state and simulates a tree of runs effected in a model by (co-)strategies. Our definition is more general than the one in [9], as we want this notion to be applicable in a broader context, including tableaux, concurrent game models and concurrent game Hintikka structures (to be defined later).

The two definitions below implicitly use the notion of descendant potential eventuality of degree  $d$  and its associate notation (see Definition 4.1). That notion was defined in the context of tableaux, however it is applicable to any CGS which is state-labelled by a set of state formulae. We recall that, given a potential eventuality  $\xi = \langle\langle A \rangle\rangle\Phi$  ( $\llbracket A \rrbracket\Phi$ ), by convention  $\xi$  itself is taken to be its (unique) descendant potential eventuality of degree 0 and that if  $\xi^i$  is a descendant eventuality of degree  $i$  of  $\xi$  then a  $\gamma$ -component of  $\xi^i$  will have the form  $\psi \wedge \langle\langle A \rangle\rangle\bigcirc \langle\langle A \rangle\rangle\Phi^{i+1}$  (respectively,  $\psi \wedge \llbracket A \rrbracket\bigcirc \llbracket A \rrbracket\Phi^{i+1}$ ) and  $\langle\langle A \rangle\rangle\Phi^{i+1}$  (respectively,  $\llbracket A \rrbracket\Phi^{i+1}$ ) will be a descendant potential eventuality of  $\xi$  having degree  $d = i + 1$ .

A piece of terminology that will be used often further: given sets  $X, Y$  and a mapping  $c : X \rightarrow Y$ , we sometimes say that *the set  $X$  is  $Y$ -coloured by  $c$*  and that for any  $x \in X$ , the value  $c(x)$  is *the  $Y$ -colour of  $x$  under the colouring  $c$* .

**Definition 5.3** (Realization witness tree for enforceable potential eventualities). Let  $\mathcal{S}$  be any (non-deterministic) CGS with a state space  $\mathbf{St}$  which is state-labelled by some set of  $\text{ATL}^+$  formulae  $\Gamma$ , with a labelling function  $c$ . Let  $s \in \mathbf{St}$  and let  $\xi \in s$  be a potential eventuality of the form  $\langle\langle A \rangle\rangle\Phi$ . A *realization witness tree for  $\xi$  at  $s$*  is a finite tree  $\mathcal{R} = (R, \rightarrow)$ , where the set of nodes  $R$  is  $\mathbf{St}$ -coloured so that:

1. the root of  $\mathcal{R}$  is coloured with  $s$  and is of depth 0;
2. if an interior node  $w$  of depth  $i$  of  $\mathcal{R}$  is coloured with  $s'$  where  $c(s') = \Theta$ , then there exists a successor  $\xi^{i+1}$  of  $\xi^i$  such that  $\langle\langle A \rangle\rangle\bigcirc \xi^{i+1} \in \Theta$ ;
3. for every interior node  $w \in \mathcal{R}$  of depth  $i$  coloured with  $s'$  where  $c(s') = \Theta$ , the children of  $w$  are coloured bijectively with vertices from an outcome set of  $\sigma_A[\langle\langle A \rangle\rangle\bigcirc \xi^i]$  at  $s'$ ;
4. if a leaf of depth  $i$  of  $\mathcal{R}$  is coloured with  $s'$  where  $c(s') = \Theta$ , then  $\xi^i = \langle\langle A \rangle\rangle\Phi \in \Theta$  is such that  $\text{Real}(\Phi, \Theta) = \text{true}$ .

**Definition 5.4** (Realization witness tree for unavoidable potential eventualities). Let  $\mathcal{S}$  be any (non-deterministic) CGS with a state space  $\mathbf{St}$  which is state-labelled by some set of  $\text{ATL}^+$  formulae  $\Gamma$ , with a labelling function  $c$ . Let  $s \in \mathbf{St}$  and let  $\xi \in s$  be a potential eventuality of the form  $\llbracket A \rrbracket\Phi$ . A *realization witness tree for  $\xi$  at  $s$*  is a finite tree  $\mathcal{R} = (R, \rightarrow)$ , where the set of nodes  $R$  is  $\mathbf{St}$ -coloured so that:

1. the root of  $\mathcal{R}$  is coloured with  $s$  and is of depth 0;

2. if an interior node  $w$  of depth  $i$  of  $\mathcal{R}$  is coloured with  $s'$  where  $c(s') = \Theta$ , then there exists a successor  $\xi^{i+1}$  of  $\xi^i$  such that  $\llbracket A \rrbracket \circ \xi^{i+1} \in \Theta$ ;
3. for every interior node  $w \in \mathcal{R}$  of depth  $i$  coloured with  $s'$  where  $c(s') = \Theta$ , the children of  $w$  are coloured bijectively with vertices from an outcome set of  $\sigma_A^c[\llbracket A \rrbracket \circ \xi^{i+1}]$  at  $s'$ ;
4. if a leaf of depth  $i$  of  $\mathcal{R}$  is coloured with  $s'$  where  $c(s') = \Theta$ , then  $\xi^i = \llbracket A \rrbracket \Phi \in \Theta$  is such that  $Real(\Phi, \Delta') = true$ .

We are going to apply the definitions above for the case when the CGS  $\mathcal{S}$  is a tableau  $\mathcal{T}_n^\eta$ , with states being (identified with) the sets of formulae in their labels.

**Lemma 5.7.** *Let  $\mathcal{R} = (R, \rightarrow)$  be a realization witness tree for a potential eventuality  $\xi$  at  $\Delta \in S_n^\eta$ . For every  $\Delta'$  of depth  $i$ , colouring a node of  $R$ ,  $\xi^i$  is realized at  $\Delta'$  in  $\mathcal{T}_n^\eta$ . In particular for  $i = 0$ , thus  $\xi$  is realized at  $\Delta$  in  $\mathcal{T}_n^\eta$ .*

*Proof.* We prove this lemma by induction on the length of the longest path from a node coloured by  $\Delta$  to a leaf of  $\mathcal{R}$ .

Base case: The length of the longest path from a node  $w$  coloured by  $\Delta$  to a leaf of  $\mathcal{R}$  is 0. Then  $w$  is a leaf and  $Real(\xi, \Delta) = true$ . Thus, by item 1 of Definition 4.2,  $\xi$  is realized at  $\Delta \in S_n^\eta$ .

Induction step: The length of the longest path from a node  $w$  coloured by  $\Delta$  to a leaf of  $\mathcal{R}$  is  $l > 0$ . Then  $w$  is an interior node of depth  $i$ , so  $\langle\langle A \rangle\rangle \xi^i \in \Delta$  (resp.  $\llbracket A \rrbracket \xi^i \in \Delta$ ) and there exists a action (resp. a co-action) such that for all children  $w'$  of  $w$ , where each  $w'$  is coloured by  $\Delta'$ ,  $\xi_\Delta^{i+1} \in \Delta'$ . Let  $\mathcal{R}'$  be a sub tree of  $\mathcal{R}$  whose root is  $w'$ . The length of the longest path from a node  $w'$  coloured by  $\Delta'$  to a leaf of  $\mathcal{R}'$  is at most  $l - 1$ . Thus, by induction hypothesis,  $\xi_\Delta^{i+1}$  is realized at  $\Delta' \in \mathcal{T}_n^\eta$  and  $\xi^j$  is realized at  $\Delta''$  in  $\mathcal{T}_n^\eta$ . Therefore  $\xi_\Delta^{i+1}$  is realized at  $\Delta'$  in  $\mathcal{T}_n^\eta$  and  $w$  respects item 2 of Definition 4.2. We conclude that  $\xi^i$  is realized at  $\Delta \in S_n^\eta$ .  $\square$

We now prove the existence of a realization witness tree for any satisfiable state of a tableau containing a potential eventuality.

**Lemma 5.8.** *Let  $\xi \in \Delta$  be a potential eventuality and  $\Delta \in S_n^\eta$  be satisfiable. Then there exists a realization witness tree  $\mathcal{R} = (R, \rightarrow)$  for  $\xi$  at  $\Delta \in S_n^\eta$ . Moreover, every  $\Delta'$ , colouring a node of  $R$ , is satisfiable.*

*Proof.* We will only give the proof for potential eventualities of the type  $\langle\langle A \rangle\rangle \Phi$ . The case of potential eventualities of type  $\llbracket A \rrbracket \Phi$  is similar.

When dealing with realization of potential eventualities, we have two cases:

1.  $Real(\Phi, \Delta) = true$ . This case is straightforward, the realization witness tree consists of only the root, coloured with  $\Delta$ .

2.  $Real(\Phi, \Delta) = false$ . This case means that there is a successor potential eventuality  $\xi_\Delta^1$  such that  $\langle\langle A \rangle\rangle \circ \xi_\Delta^1 \in \Delta$ .

As  $\Delta$  is satisfiable, there exists a CGM  $\mathcal{M}$  and a state  $s \in \mathcal{M}$  such that  $\mathcal{M}, s \models \Delta$ , and in particular,  $\mathcal{M}, s \models \langle\langle A \rangle\rangle \circ \xi_\Delta^1$ . Thus, there exists an  $A$ -action  $\sigma_A \in \text{act}_A^c(s)$  such that  $\mathcal{M}, s' \models \xi_\Delta^1$  for all  $s' \in \text{Out}(s, \sigma_A)$ , that is an  $A$ -action witnessing the truth of  $\langle\langle A \rangle\rangle \circ \xi_\Delta^1$  at  $s$ .

We know that  $\Delta$  is satisfiable and that  $\langle\langle A \rangle\rangle \circ \xi_\Delta^1$  is an enforceable successor formula. Let  $p$  be the position of  $\langle\langle A \rangle\rangle \circ \xi_\Delta^1$  in the list made at the application of the rule **(Next)** on  $\Delta$ . Note that  $\xi$  is a  $\gamma$ -formula  $\in cl(\eta)$ , where at least one of its  $\gamma$ -components, obtained from a pair  $\langle\psi, \Psi\rangle$ , is such that  $Real(\Psi, FE(\psi)) = true$ . Let  $\chi$  be such a  $\gamma$ -component. So Lemma 5.5 is applicable to  $\Delta$ , and according to that corollary, there exists an outcome set  $X_\chi$  of  $\sigma_A[\langle\langle A \rangle\rangle \circ \xi_\Delta^1]$  at  $\Delta$  such that, for every  $\Delta' \in X_\chi$ , there exists  $s' \in \text{Out}(s, \sigma)$  such that  $\mathcal{M}, s' \models \Delta'$ , and moreover, if  $\mathcal{M}, s' \models \chi$ , then  $\chi \in \Delta'$ . We start building the realization witness tree  $\mathcal{R}$  with a simple tree whose root  $r$  is coloured with  $\Delta$  and whose leaves are coloured bijectively with a node from  $X$ . This first tree respects Items 1 to 3 of Definition 5.3; some of the leaves respect Item 4 of this definition, but not all of them. The next part treats these leaves.

Since,  $\mathcal{M}, s \models \xi_\Delta^1$  with  $\xi_\Delta^1 = \langle\langle A \rangle\rangle \Phi'$  for every  $s' \in \text{Out}(s, \sigma_A)$ , it follows that for every such  $s'$  there exists a perfect-recall  $A$ -strategy  $F_A^{s'}$  such that for every  $\lambda \in \text{Plays}(s', F_A^{s'})$ ,  $\mathcal{M}, \lambda \models \Phi'$ . Then, playing  $\sigma_A$  followed by playing  $F_A^{s'}$  constitutes a perfect recall strategy  $F_A$  witnessing the truth of  $\xi$  at  $s$ .

Then we continue the construction of  $\mathcal{R}$  as follows. Let  $S'$  be the set of all states  $s''$  appearing as part of a play consistent with  $F_A^{s'}$ , containing a descendant eventuality  $\xi^i$  of  $\xi$  and satisfying the requirement that  $\mathcal{M}, s'' \not\models \chi$ , for all  $\gamma$ -components  $\chi$  obtained from a pair  $\langle\psi, \Psi\rangle$  such that  $Real(\Psi, FE(\psi)) = true$ . For every  $s' \in \text{Out}(s, \sigma_A)$ , we follow the perfect recall strategy  $F_A^{s'}$ , matching every state  $s'' \in S'$  with a node  $w''$  of  $\mathcal{R}$  and matching every  $A'$ -action of  $F_A^{s'}$  at  $s''$  with the tableau  $\sigma_A[\langle\langle A \rangle\rangle \circ \xi^i] \in \text{act}_A(\Delta'')$  where  $\xi^i$  is the descendant eventuality of  $\xi$  in  $w''$  and  $\Delta''$  is the state colouring the node  $w''$ . We follow this way each  $F_A^{s'}$  along each run until we reach a state  $t$  where  $\mathcal{M}, t \models \chi$  ( $\chi$  is as described above). This means that we have reached a leaf of  $\mathcal{R}$ ; this leaf respects item 4 of Definition 5.3. As  $\mathcal{M}, s \models \xi$ , such a state can be reached for each run, and we thus obtain a finite tree  $\mathcal{R}$ .

Thus, the so constructed realization witness tree conforms to Definition 5.3.  $\square$

$\square$

**Lemma 5.9.** *Let  $\Delta$  be a state in  $\mathcal{T}_n^\eta$ . If  $\Delta$  is satisfiable then Rule (ER2) cannot eliminate  $\Delta$  from  $\mathcal{T}_n^\eta$ .*

*Proof.* Let  $\Delta \in \mathcal{T}_n^\eta$  be a satisfiable state.

If  $\Delta$  contains no eventuality, then Rule **(ER2)** is not applicable.

If  $\Delta$  contains an eventuality  $\xi$ , then Lemma 5.8 ensures that there exists a realization witness tree for  $\Delta$  and, by Lemma 5.7 we know that  $\xi$  is realized at  $\Delta$  in  $\mathcal{T}_n^\eta$ . Therefore, Rule **(ER2)** cannot eliminate  $\Delta$  for  $\mathcal{T}_n^\eta$ .  $\square$   $\square$

**Theorem 5.2** (Soundness). *If  $\eta$  is satisfiable, then  $\mathcal{T}^\eta$  is open.*

*Proof.* Lemmas 5.3–5.9 ensure that if  $\Delta$  is satisfiable, then  $\Delta$  cannot be eliminated from  $\mathcal{T}_n^\eta$  due to Rule **(ER1)** or Rule **(ER2)**. Moreover, Lemma 5.1 ensures that if the input formula  $\eta$  is satisfiable, then at least one state containing  $\eta$  (created from the initial prestate) is satisfiable. Thus, this state cannot be eliminated and therefore the final tableau  $\mathcal{T}^\eta$  is open.  $\square$   $\square$

## 6 Completeness, model synthesis and complexity

### 6.1 Hintikka Structures

The tableau procedure actually attempts to build not a concurrent game model of the input formula but a state-labelled non-deterministic CGS, from which structures of a special kind can be extracted which essentially are partly defined concurrent game models. Following [11, 2, 9] we will call them *Hintikka structures*. Here we will give the definition of a Hintikka structure for a given  $\text{ATL}^+$  formula  $\eta$  and will show how to obtain a CGM for  $\eta$  from a Hintikka structure for  $\eta$ . Later we will explain how to extract a Hintikka structure ‘satisfying’ the input formula from its open final tableau.

**Definition 6.1.** A *Concurrent Game Hintikka Structure* (for short, CGHS) is a deterministic CGS  $\mathcal{H} = (\mathbb{A}, \text{St}, \{\text{Act}_a\}_{a \in \mathbb{A}}, \{\text{act}_a\}_{a \in \mathbb{A}}, \text{out}, H)$  which is state-labelled by a given set  $\Gamma$  of  $\text{ATL}^+$ -formulae with a state-labelling function  $H$ . Let  $s \in \text{St}$  be a state of  $\mathcal{H}$ . An Hintikka structure  $\mathcal{H}$  satisfies the following constraints:

- H1** If  $\varphi \in H(s)$  then  $\neg\varphi \notin H(s)$ ;
- H2** If an  $\alpha$ -formula belongs to  $H(s)$ , then its both  $\alpha$ -components do;
- H3** If a  $\beta$ -formula belongs to  $H(s)$ , then one of its  $\beta$ -components does;
- H4** If a  $\gamma$ -formula belongs to  $H(s)$ , then one of its  $\gamma$ -components does;
- H5** If  $\langle\langle A \rangle\rangle\psi \in H(s)$ , then there exists an  $A$ -action  $\sigma_A \in \text{act}_A(s)$  such that  $\psi \in H(s')$  for all  $s' \in \text{Out}(s, \sigma_A)$ . Likewise, if  $\llbracket A \rrbracket\psi \in H(s)$ , then there exists an  $A$ -co-action  $\sigma_A^c \in \text{act}_A^c(s)$  such that, for all  $\psi \in H(s')$  for all  $s' \in \text{Out}(s, \sigma_A^c)$ .



**H6** If a potential eventuality  $\xi = \langle\langle A \rangle\rangle\Phi$  (resp.  $\xi = \llbracket A \rrbracket\Phi$ ) belongs to  $H(s)$ , then there exists a realization witness tree, rooted at  $s$  in  $\mathcal{H}$  for  $\xi = \langle\langle A \rangle\rangle\Phi$  (resp.  $\xi = \llbracket A \rrbracket\Phi$ ) at  $s$ .

*Remark 6.1.* The condition H6 is well defined because a Hintikka structure is obtained by colouring via  $H$  from a deterministic concurrent game structure, for which the notion of realization witness tree is defined.

**Definition 6.2.** Let  $\mathcal{H} = (\mathbb{A}, \text{St}, \{\text{Act}_a\}_{a \in \mathbb{A}}, \{\text{act}_a\}_{a \in \mathbb{A}}, \text{out}, H)$  be a CGHS and  $\eta$  be an  $\text{ATL}^+$ -formula. We say that  $\mathcal{H}$  is a concurrent game Hintikka structure for  $\eta$ , if  $\eta \in H(s)$  for some  $s \in \text{St}$ .

We now show that from any CGHS for a given formula  $\eta$  a CGM satisfying  $\eta$  can be obtained.

**Theorem 6.1.** Let  $\mathcal{H} = (\mathbb{A}, \text{St}, \{\text{Act}_a\}_{a \in \mathbb{A}}, \{\text{act}_a\}_{a \in \mathbb{A}}, \text{out}, H)$  be a CGHS for a given  $\text{ATL}^+$ -formula  $\eta$ . Let further  $\mathcal{M} = (\mathbb{A}, \text{St}, \{\text{Act}_a\}_{a \in \mathbb{A}}, \{\text{act}_a\}_{a \in \mathbb{A}}, \text{out}, \text{Prop}, \text{L})$  be the CGM obtained from  $\mathcal{H}$  by setting, for every  $s \in \text{St}$ ,  $\text{L}(s) = H(s) \cap \text{Prop}$ . Then, for every  $s \in \text{St}$  and every  $\text{ATL}^+$  formula  $\varphi$ ,  $\varphi \in H(s)$  implies  $\mathcal{M}, s \models \varphi$ . In particular,  $\mathcal{M}$  satisfies  $\eta$ .

*Proof.* Suppose  $\varphi \in H(s)$ . We will prove that  $\mathcal{M}, s \models \varphi$  by induction on the structure of the state formula  $\varphi$ .

*Base.* If  $\varphi \in \text{Prop} \cup \{\top\}$  belongs to  $H(s)$ , it is immediate that  $\mathcal{M}, s \models \varphi$ , by definition of  $\text{L}$  and H1.

*Inductive Step.*

- $\varphi$  is  $\psi_1 \wedge \psi_2$ . By H2 we get that  $\psi_1 \in H(s)$  and  $\psi_2 \in H(s)$ . By inductive hypothesis  $\mathcal{M}, s \models \psi_1$  and  $\mathcal{M}, s \models \psi_2$ . Therefore  $\mathcal{M}, s \models \varphi$ .
- $\varphi$  is  $\psi_1 \vee \psi_2$ . By H3 we get that either  $\psi_1 \in H(s)$  or  $\psi_2 \in H(s)$ . By inductive hypothesis either  $\mathcal{M}, s \models \psi_1$  or  $\mathcal{M}, s \models \psi_2$ . Therefore  $\mathcal{M}, s \models \varphi$ .
- $\varphi$  is  $\langle\langle A \rangle\rangle\psi$  or  $\llbracket A \rrbracket\psi$ . An application of H5 and the inductive hypothesis to  $\psi$  imply that  $\mathcal{M}, s \models \varphi$ .
- $\varphi$  is  $\langle\langle A \rangle\rangle\Phi$  or  $\varphi$  is  $\llbracket A \rrbracket\Phi$ , where  $\Phi$  is a path formula whose main operator is different from  $\bigcirc$ , that is  $\varphi$  is a  $\gamma$ -formula. Here we only present in detail the first case, the second one being quite similar. We need to prove the existence of a (perfect recall) strategy  $F_A$  such that, for each branch  $\lambda$  in  $\mathcal{M}$  stemming from  $s$  and consistent with that strategy,  $\mathcal{M}, \lambda \models \Phi$ . This will imply that  $\mathcal{M}, s \models \varphi$ . Since  $\varphi = \langle\langle A \rangle\rangle\Phi \in H(s)$  by hypothesis, then H6 guarantees the existence of a realization witness tree  $T$  on  $\mathcal{H}$  for  $\varphi$ . By construction,  $T$  provides a partial finite strategy  $Fp_A$ , defined only for the finite set of histories occurring in  $T$  and having length strictly less than the height of  $T$ . We want to show that  $Fp_A$  can be extended to a strategy  $F_A$ , defined for all the histories in  $\mathcal{H}$ , and that  $T$  can be extended to a possibly infinite tree  $T'$  such that:

- Each node of  $T'$  is also a node of  $\mathcal{H}$  and each labelled edge of  $T'$  is also a labelled edge of  $\mathcal{H}$ .
- All paths in  $T'$  are consistent with  $F_A$ , hence  $T'$  witnesses the truth of  $\langle\langle A \rangle\rangle\Phi$  at state  $s$  of  $\mathcal{M}$  by instantiating the quantifier  $\langle\langle A \rangle\rangle$  to  $F_A$ .

Below, we show how to construct  $F_A$  and  $T'$ . Let us consider any finite path in  $T$  of the form  $\lambda_{\leq n}$ , where  $\lambda_0 = s$  and  $\lambda_n$  is a leaf. By construction of  $T$ , each node  $\lambda_i$ , for  $1 \leq i \leq n$ , is a node of  $\mathcal{H}$  and each labelled edge of  $T$  is a labelled edge of  $\mathcal{H}$ . The descendant potential eventuality  $\varphi^n$  of  $\varphi$  belongs to the colour of  $\lambda_n$  by construction of  $T$ . Since  $\lambda_n$  is a node of  $\mathcal{H}$  and  $\varphi^n \in H(\lambda_n)$ , by H4 some  $\gamma$ -component  $\chi$  of  $\varphi^n$  belongs to  $H(\lambda_n)$ . This formula  $\chi$  is either of the form  $\psi$  or of the form  $\psi \wedge \langle\langle A \rangle\rangle \bigcirc \varphi^{n+1}$  (the second case occurs, for instance, when  $\varphi$  has the form  $\langle\langle A \rangle\rangle \Box \theta$ ).

In the first case, any extension of the partial strategy  $Fp_A$  and any extension of  $\lambda_{\leq n}$  to an infinite path will do.

In the second case, we apply H2 to get  $\psi \in H(\lambda_n)$  and  $\langle\langle A \rangle\rangle \bigcirc \varphi^{n+1} \in H(\lambda_n)$ . By H5, there exists an A-action  $\sigma_A \in \text{act}_A(H(\lambda_n))$  such that  $\varphi^{n+1} \in H(s')$  for all  $s' \in \text{Out}(\lambda_n, \sigma_A)$ . Playing this A-action  $\sigma_A$  after the partial strategy  $Fp_A$  gives us a new partial strategy  $Fp'_A$  defined for histories whose length is less than or equal to  $n$ . The set of successors of  $\lambda_n$  for  $T'$  is the set  $\text{Out}(\lambda_n, \sigma_A)$ . For each  $s' \in \text{Out}(\lambda_n, \sigma_A)$ , we can again apply H2, H4 and H5 to get a new partial strategy  $Fp''_A$  defined for histories whose length is inferior or equal to  $n + 1$ . For any  $s' \in \text{Out}(\lambda_n, \sigma_A)$ , its successors are obtained by an application of  $Fp''_A$ . An infinite iteration of this procedure will give us the complete strategy  $F_A$  and the way to extend the finite tree so as to get  $T'$ . □

## 6.2 Completeness and model synthesis

**Theorem 6.2.** *The tableau method for  $\text{ATL}^+$  is complete.*

Completeness of the procedure means that an open tableau for  $\eta$  implies existence of a CGM model for  $\eta$ . So, we start with an open tableau  $\mathcal{T}^\eta$  for  $\eta$  and we want to prove that  $\eta$  is indeed satisfiable. The proof is constructive, as we will build from  $\mathcal{T}^\eta$  a Hintikka structure  $\mathcal{H}_\eta$  that can be turned into a model for  $\eta$ . In order to construct that Hintikka structure, first we will extract special  $\mathcal{T}^\eta$ -trees associated with potential eventualities, that can be seen as building modules to be used to construct the entire structure. Eventually, we show that the so constructed structure is a Hintikka structure for  $\eta$ .

First, we need to define *edge-labelling* of a tree.

**Definition 6.3.** Let  $\mathcal{W} = (W, \rightsquigarrow)$  be a tree and  $Y$  be a non-empty set. An *edge-labelling of  $\mathcal{W}$  by  $Y$*  is a mapping  $l$  from the set of edges of  $\mathcal{W}$  to the set of non-empty subsets of  $Y$ .

**Definition 6.4.** Given a tableau  $\mathcal{T}^\eta$ , a tree  $\mathcal{W} = (W, \rightsquigarrow)$  is a  $\mathcal{T}^\eta$ -tree if the following conditions hold:

- $\mathcal{W}$  is  $S^\eta$ -coloured, by some colouring mapping  $c$ .
- $\mathcal{W}$  is edge-labelled by  $\bigcup_{(\Delta \in S^\eta)} \text{act}_\Delta(\Delta)$ , by some edge-labelling mapping  $l$ ;
- $l(w \rightsquigarrow w') \subseteq \text{act}_\Delta(\Delta)$  for every  $w \in W$  with  $c(w) = \Delta$ ;
- For every interior node  $w \in W$  with  $c(w) = \Delta$  and every successor  $\Delta' \in \mathcal{T}^\eta$  of  $\Delta$ , there exists exactly one  $w' \in W$  such that  $l(w \rightsquigarrow w') = \{\sigma \mid \Delta \xrightarrow{\sigma} \Delta'\}$ .

**Definition 6.5.** Let  $\Delta \in S^\eta$ . A  $\mathcal{T}^\eta$ -tree  $\mathcal{W}$  is *rooted at*  $\Delta$  if the root  $r$  of  $\mathcal{W}$  is coloured with  $\Delta$ .

For the purpose of our construction, we distinguish two kinds of  $\mathcal{T}^\eta$ -trees: *simple* or *realizing*. Their definitions are given below. Realizing  $\mathcal{T}^\eta$ -trees will deal especially with potential eventualities.

**Definition 6.6.** A tree  $\mathcal{W} = (W, \rightsquigarrow)$  is *simple* if it has no interior nodes except the root.

Simple  $\mathcal{T}^\eta$ -trees can be seen as one-step modules.

**Definition 6.7.** Let  $\mathcal{W} = (W, \rightsquigarrow)$  be a  $\mathcal{T}^\eta$ -tree rooted at  $\Delta$  and  $\xi \in \Delta$  a potential eventuality. The tree  $\mathcal{W}$  is a *realizing  $\mathcal{T}^\eta$ -tree for*  $\xi$ , denoted  $\mathcal{W}_\xi$ , if there exists a subtree  $\mathcal{R}_\xi$  of  $\mathcal{W}$  rooted at  $\Delta$  such that  $\mathcal{R}_\xi$  is a realization witness tree for  $\xi$  rooted at  $\Delta \in \mathcal{T}^\eta$ .

**Lemma 6.1.** *Let  $\Delta \in S^\eta$ . Then, there exists a simple  $\mathcal{T}^\eta$ -tree rooted at  $\Delta$ .*

*Proof.* We construct a simple  $\mathcal{T}^\eta$ -tree  $\mathcal{W}$  rooted at  $\Delta$  as follows. The root of  $\mathcal{W}$  is a node  $r$  such that  $c(r) = \Delta$ . For every successor state  $\Delta'$  of  $\Delta \in \mathcal{T}^\eta$ , let  $Moves(\Delta, \Delta') = \{\sigma \mid \Delta \xrightarrow{\sigma} \Delta'\}$ . Note that, by construction of the tableau, the family  $\{Moves(\Delta, \Delta') \mid \Delta' \text{ is a successor of } \Delta\}$  is a partition of the set  $\text{act}_\Delta(\Delta)$  of all action profiles applied at  $\Delta$ . Now, for each set  $X$  of that family we select one successor  $\Delta'$  of  $\Delta$  such that  $X = \text{act}(\Delta, \Delta')$  and add a successor  $t$  to  $\mathcal{W}$  such that  $c(t) = \Delta'$  and  $l(r \rightsquigarrow t) = \{\sigma \mid \Delta \xrightarrow{\sigma} \Delta'\}$ .  $\square$   $\square$

**Example 6.1.** (*Continuation of Example 4.7*)

Consider the final tableau  $\mathcal{T}^\vartheta$  for the formula  $\vartheta = \langle\langle 1 \rangle\rangle(p\mathcal{U}q \vee \Box q) \wedge \llbracket 2 \rrbracket(\Diamond p \wedge \Box \neg q)$ .

We have seen in the example 4.7 that  $S^\vartheta = \{\Delta_1, \dots, \Delta_8\}$ .

We have listed possible simple  $\mathcal{T}^\vartheta$ -trees rooted at each  $\Delta_i$  in the table on Figure 10.

To show the existence of a realizing  $\mathcal{T}^\eta$ -tree for  $\xi$  at  $\Delta$ , we first prove the existence of a realization witness tree  $\mathcal{R}_\xi$  for  $\xi$  at  $\Delta$ .

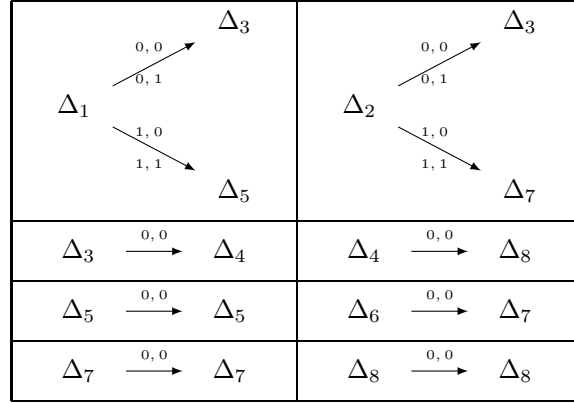


Figure 10: Simple  $\mathcal{T}^\theta$ -trees

**Lemma 6.2.** *Let  $\mathcal{T}^\eta$  be a tableau for  $\eta$  and  $\xi$  be a potential eventuality realized at  $\Delta \in \mathcal{T}^\eta$ . Then, there exists a realization witness tree  $\mathcal{R}_\xi$  for  $\xi$  at  $\Delta$  in  $\mathcal{T}^\eta$ .*

*Proof.* We give detailed proof only for the case where  $\xi = \langle\langle A \rangle\rangle\Phi$ ; the other case is similar, just replace  $\langle\langle A \rangle\rangle$  by  $\llbracket A \rrbracket$  in the proof. Suppose that  $\xi$  is realized at  $\Delta$  in  $\mathcal{T}^\eta$ . We define the *rank* of  $\xi$  at  $\Delta$  in  $\mathcal{T}^\eta$ , denoted  $\text{rank}(\xi, \Delta, \mathcal{T}^\eta)$  to be the minimal length of a chain of descendant potential eventualities  $\xi = \xi^0, \dots, \xi^n = \langle\langle A \rangle\rangle\Phi_n$  ensuring the realization of  $\xi$ , that is,  $\text{Real}(\Phi_n, \Delta_j) = \text{true}$  for some state  $\Delta_j$  descendant of  $\Delta$  in  $\mathcal{T}^\eta$ . We prove the existence of a realization witness tree  $\mathcal{R}$  for  $\xi$  at  $\Delta$  in  $\mathcal{T}^\eta$  by induction on rank  $\text{rank}(\xi, \Delta, \mathcal{T}^\eta)$ .

Base:  $\text{rank}(\xi, \Delta, \mathcal{T}^\eta) = 0$ . Here  $\xi$  is immediately realized and  $\mathcal{R}_\xi$  contains only the root coloured with  $\Delta$ .

Inductive step:  $\text{rank}(\xi, \Delta, \mathcal{T}^\eta) = k$  where  $k > 0$ . Since  $\xi$  is realized at  $\Delta \in \mathcal{T}^\eta$  and  $\text{rank}(\xi, \Delta, \mathcal{T}^\eta) > 0$ , by Definition 4.2 we have that for every  $\sigma \in D(\Delta, \langle\langle A \rangle\rangle\Phi^1)$  there exists  $\Delta' \in \mathcal{T}^\eta$  such that  $\Delta \xrightarrow{\sigma} \Delta'$  and  $\xi^1$  is realized at  $\Delta' \in \mathcal{T}^\eta$ . We build a tree  $T$  rooted at a node  $r$  coloured with  $\Delta$  where the children  $v$  of  $r$  are coloured bijectively with the set of  $\Delta'$  obtained above. Then  $\text{rank}(\xi^1, \Delta', \mathcal{T}^\eta) = k - 1$  and we can apply the inductive hypothesis to obtain a realization witness tree  $\mathcal{R}_{\xi^1}$  for  $\xi^1$  at  $\Delta'$  in  $\mathcal{T}^\eta$  for each  $\Delta'$ . Thus, replacing each node  $v$  of  $T$  by the corresponding  $\mathcal{R}_{\xi^1}$  gives us  $\mathcal{R}_\xi$ .  $\square$

**Lemma 6.3.** *Let  $\xi \in \Delta \in S^\eta$  be a potential eventuality. Then, there exists a finite realizing  $\mathcal{T}^\eta$ -tree for  $\xi$  rooted at  $\Delta$ .*

*Proof.* Since  $\mathcal{T}^\eta$  is open,  $\xi$  is realized at  $\Delta$  in  $\mathcal{T}^\eta$ . To construct the realizing  $\mathcal{T}^\eta$ -tree  $\mathcal{W}_\xi$  for  $\xi$  rooted at  $\Delta$ , we start from the realization witness tree  $\mathcal{R}_\xi$ , whose existence is given by Lemma 6.2 and provisionally we take  $\mathcal{W}_\xi$  to be  $\mathcal{R}_\xi$ . The problem with  $\mathcal{R}_\xi$  is that for some  $\sigma \in \text{act}_\mathbb{A}(\Delta)$  at some node  $w$  of  $\mathcal{R}_\xi$ , there is no edge  $w \rightsquigarrow w'$  such that  $l(w \rightsquigarrow w') \ni \sigma$ . Therefore, to extend  $\mathcal{W}_\xi$  into a realizing  $\mathcal{T}^\eta$ -tree, for every such node  $w$ , we pick one of the successor

states of  $c(w)$  via  $\sigma$ , say  $\Delta'$  and add a node  $w'$  to  $\mathcal{W}_\xi$  such that  $c(w') = \Delta'$  and  $l(w \rightsquigarrow w') \ni \sigma$ .  $\square$   $\square$

**Example 6.2.** (*Continuation of Example 6.1*) We now extract in Figure 11 a possible realizing tree from the open tableau for every state  $\Delta_i$ , using realization witness trees.

We now construct a final structure, denoted by  $\mathfrak{F}$ , from simple and realizing  $\mathcal{T}^\eta$ -trees. This construction is made step-by-step. At the end of the construction, we prove that  $\mathfrak{F}$  is indeed a Hintikka structure.

**Step 1.** We define a grid  $\mathcal{F}$  of size  $m \times n$ , where  $m$  is the number of eventualities occurring in  $\mathcal{T}^\eta$  and  $n$  the number of states of  $\mathcal{T}^\eta$ . Each row of that grid is labelled by one of the potential eventualities and each column by a state of  $\mathcal{T}^\eta$  previously ordered by name ( $\Delta_i < \Delta_j$  if  $i < j$ ). We denote by  $\xi_i$  the eventuality associated to row  $0 \leq i \leq m$ , we denote by  $\Delta_j$  the state associated to the column  $0 \leq j \leq n$ . The content  $\mathcal{F}(i, j)$  of each intersection between a row  $i$  and a column  $j$  of  $\mathcal{F}$  is as follows: if  $\xi_i \in \Delta_j$ , then  $\mathcal{F}(i, j)$  is the realizing  $\mathcal{T}^\eta$ -tree for  $\xi_i$  rooted at  $\Delta_j$ , whose existence is ensured by Lemma 6.3; otherwise,  $\mathcal{F}(i, j)$  is the simple  $\mathcal{T}^\eta$ -tree rooted at  $\Delta_j$ , whose existence is ensured by Lemma 6.1.

**Example 6.3.** (*Continuation of Example 6.2*) The grid  $\mathcal{F}$  for our example has a size  $4 \times 8$  and is represented in Figure 12.

**Step 2.** We make a queue  $\mathcal{Q}$  that will contain potential eventualities occurring in  $\mathcal{T}^\eta$ . The first element of  $\mathcal{Q}$  is either  $\eta$ , if  $\eta$  is a potential eventuality, or the potential eventuality associated to the first column of the grid defined just above. Let  $\xi_i$  be the first element of the queue, so that  $\mathcal{Q}(0) = \xi_i$ . Then we add to  $\mathcal{Q}$  all the potential eventualities following the order of grid's rows and cycling if necessary, that is  $\mathcal{Q}(k) = \xi_{((i+k) \bmod m)}$  for  $k \in [1, m-1]$ .

**Step 3.** Let  $\Delta$  be one of the states containing  $\eta$ . Next, we take the element  $\mathcal{F}(\mathcal{Q}(0), \Delta)$  of the grid. The root of  $\mathcal{F}(\mathcal{Q}(0), \Delta)$  is then the root of  $\mathfrak{F}$ . Then we take one-by-one in order all the elements of the rest of the queue and do the following:

Let  $\mathcal{Q}(i)$  be the current element of the queue to be treated. For every dead-end state  $w \in \mathfrak{F}$ , that is a state without successors, such that  $c(w) = \Delta_j$ , we add the tree  $\mathcal{F}(\mathcal{Q}(i), \Delta_j)$  by identifying the dead-end state  $w$  with the root of  $\mathcal{F}(\mathcal{Q}(i), \Delta_j)$ ;

**Example 6.4.** From the grid  $\mathcal{F}$ , we can extract in four steps a partial structure  $\mathfrak{F}$  realizing all the eventualities (see Figure 13).

**Step 4.** Finally, we ensure that  $\mathfrak{F}$  finite. While there is a dead-end state in  $\mathfrak{F}$ , say  $w$  with  $c(w) = \Delta_j$ , we choose a component from the row  $\mathcal{F}(\Delta_j)$  as follows:

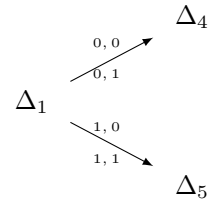
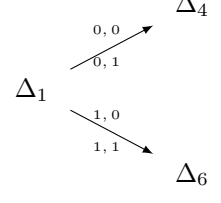
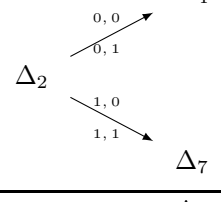
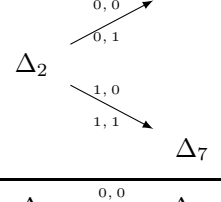
State	Eventualities	Rank	Realization Witness Tree	Realizing Tree
$\Delta_1$	$\xi_1 = \langle\langle 1 \rangle\rangle(p\mathcal{U}q \vee \Box q)$	1	$\Delta_1 \xrightarrow[0,1]{0,0} \Delta_4$	
$\Delta_1$	$\xi_2 = \llbracket 2 \rrbracket(\Diamond p \wedge \Box \neg q)$	1	$\Delta_1 \xrightarrow[1,1]{1,0} \Delta_6$	
$\Delta_2$	$\xi_1 = \langle\langle 1 \rangle\rangle(p\mathcal{U}q \vee \Box q)$	1	$\Delta_2 \xrightarrow[0,1]{0,0} \Delta_4$	
$\Delta_2$	$\xi_2 = \llbracket 2 \rrbracket(\Diamond p \wedge \Box \neg q)$	1	$\Delta_2 \xrightarrow[1,1]{1,0} \Delta_7$	
$\Delta_3$	$\xi_3 = \xi_1^1 = \langle\langle 1 \rangle\rangle p\mathcal{U}q$	1	$\Delta_3 \xrightarrow{0,0} \Delta_4$	$\Delta_3 \xrightarrow{0,0} \Delta_4$
$\Delta_4$	$\xi_3 = \xi_1^1 = \langle\langle 1 \rangle\rangle p\mathcal{U}q$	0	$\Delta_4$	$\Delta_4 \xrightarrow{0,0} \Delta_8$
$\Delta_5$	$\xi_2 = \llbracket 2 \rrbracket(\Diamond p \wedge \Box \neg q)$	1	$\Delta_5 \xrightarrow{0,0} \Delta_6$	$\Delta_5 \xrightarrow{0,0} \Delta_6$
$\Delta_6$	$\xi_2 = \llbracket 2 \rrbracket(\Diamond p \wedge \Box \neg q)$	0	$\Delta_6$	$\Delta_6 \xrightarrow{0,0} \Delta_7$
$\Delta_7$	$\xi_4 = \xi_2^1 = \llbracket 2 \rrbracket \Box \neg q$	0	$\Delta_7$	$\Delta_7 \xrightarrow{0,0} \Delta_7$

Figure 11: Eventualities and realizing  $\mathcal{T}^\vartheta$ -trees

$\mathcal{F}$		0	1	2	3
		$\xi_1$	$\xi_2$	$\xi_3$	$\xi_4$
0	$\Delta_1$				
1	$\Delta_2$				
2	$\Delta_3$	$\Delta_3 \xrightarrow{0,0} \Delta_4$	$\Delta_3 \xrightarrow{0,0} \Delta_4$	$\Delta_3 \xrightarrow{0,0} \Delta_4$	$\Delta_3 \xrightarrow{0,0} \Delta_4$
3	$\Delta_4$	$\Delta_4 \xrightarrow{0,0} \Delta_8$	$\Delta_4 \xrightarrow{0,0} \Delta_8$	$\Delta_4 \xrightarrow{0,0} \Delta_8$	$\Delta_4 \xrightarrow{0,0} \Delta_8$
4	$\Delta_5$	$\Delta_5 \xrightarrow{0,0} \Delta_5$	$\Delta_5 \xrightarrow{0,0} \Delta_6$	$\Delta_5 \xrightarrow{0,0} \Delta_5$	$\Delta_5 \xrightarrow{0,0} \Delta_5$
5	$\Delta_6$	$\Delta_6 \xrightarrow{0,0} \Delta_7$	$\Delta_6 \xrightarrow{0,0} \Delta_7$	$\Delta_6 \xrightarrow{0,0} \Delta_7$	$\Delta_6 \xrightarrow{0,0} \Delta_7$
6	$\Delta_7$	$\Delta_7 \xrightarrow{0,0} \Delta_7$	$\Delta_7 \xrightarrow{0,0} \Delta_7$	$\Delta_7 \xrightarrow{0,0} \Delta_7$	$\Delta_7 \xrightarrow{0,0} \Delta_7$
7	$\Delta_8$	$\Delta_8 \xrightarrow{0,0} \Delta_8$	$\Delta_8 \xrightarrow{0,0} \Delta_8$	$\Delta_8 \xrightarrow{0,0} \Delta_8$	$\Delta_8 \xrightarrow{0,0} \Delta_8$

Figure 12: The grid  $\mathcal{F}$

Eventualities	Added grid elements	Partial structure $\widehat{\mathfrak{F}}$
$\{\xi_1, \xi_2, \xi_3, \xi_4\}$	$\mathcal{F}(\xi_1, \Delta_1)$	
$\{\xi_2, \xi_3, \xi_4\}$	$\mathcal{F}(\xi_2, \Delta_4); \mathcal{F}(\xi_2, \Delta_5)$	
$\{\xi_3, \xi_4\}$	$\mathcal{F}(\xi_3, \Delta_8); \mathcal{F}(\xi_3, \Delta_6)$	
$\{\xi_4\}$	$\mathcal{F}(\xi_4, \Delta_8); \mathcal{F}(\xi_4, \Delta_8)$	

Figure 13: Eventualities and partial structures



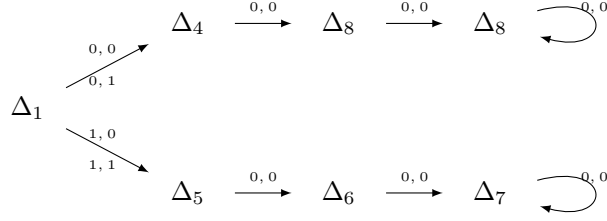


Figure 14: The final structure  $\mathfrak{F}$  for the formula  $\vartheta = \langle\langle 1 \rangle\rangle(p\mathcal{U}q \vee \Box q) \wedge \llbracket 2 \rrbracket(\Diamond p \wedge \Box \neg q)$

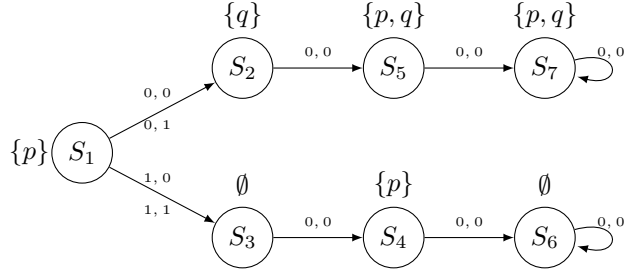


Figure 15: Concurrent game model satisfying  $\vartheta$

- With priority we choose a component  $\mathcal{F}(i, \Delta_j), 0 \leq i \leq m$  already occurring in  $\mathfrak{F}$ . Let  $r$  be the root of the component  $\mathcal{F}(i, \Delta_j), 0 \leq i \leq m$  inside  $\mathfrak{F}$ . Then we add an arrow  $\rightsquigarrow$  between every predecessor  $v$  of  $w$  and the root  $r$  and labelled this arrow with  $l(v \rightsquigarrow w)$ . Then we delete the node  $w \in \mathfrak{F}$ .
- Otherwise, if the chosen component  $\mathcal{F}(i, \Delta_j)$  is not already occurring in  $\mathfrak{F}$  then we add the new component to  $\mathfrak{F}$  as usual by merging the root of the component with the dead-end state  $w$ .

When there are no longer dead-ends in  $\mathfrak{F}$ , the structure is completed and we have obtained our final structure.

**Example 6.5.** The final structure  $\mathfrak{F}$  for the formula  $\vartheta = \langle\langle 1 \rangle\rangle(p\mathcal{U}q \vee \Box q) \wedge \llbracket 2 \rrbracket(\Diamond p \wedge \Box \neg q)$  is given in Figure 14.

By keeping only the propositional variables in the state labels we obtain the following concurrent game model satisfying  $\vartheta$  in Figure 15.

**Lemma 6.4.** *Let  $\mathcal{T}$  be a  $\mathcal{T}^\eta$ -tree rooted at  $\Delta = c(w)$ . Then, the following holds:*

1. *If  $\langle\langle A \rangle\rangle \bigcirc \varphi \in \Delta$ , then there exists an  $A$ -action  $\sigma_A \in \text{act}_A(\Delta)$  such that  $\varphi \in c(w') = \Delta'$  where  $l(w \rightsquigarrow w') \ni \sigma$  for every  $\sigma \sqsupseteq \sigma_A$ .*

2. If  $\llbracket A \rrbracket \bigcirc \varphi \in \Delta$  then there exists a co- $A$ -action  $\sigma_A^c \in \text{act}_A^c(\Delta)$  such that  $\varphi \in c(w') = \Delta'$  where  $l(w \rightsquigarrow w') \ni \sigma$  for every  $\sigma \sqsupseteq \sigma_A^c(\sigma_A)$ .

*Proof.* We recall that all successor formulae of  $\Delta \in S^\eta$  are ordered at the application of the rule **(Next)** to  $\Delta$ .

(1) Suppose that  $\langle\langle A \rangle\rangle \bigcirc \varphi \in \Delta$ . Then the required  $A$ -action is  $\sigma_A[\langle\langle A \rangle\rangle \bigcirc \varphi]$ . Indeed, it immediately follows from the rule **(Next)** that for every  $\sigma \sqsupseteq \sigma_A$  in the pretableau  $\mathcal{P}^\eta$ , if  $\Delta \xrightarrow{\sigma} \Gamma$ , then  $\varphi \in \Gamma$  and  $\varphi \in \Delta'$  since  $\Delta'$  is a full expansion of  $\Gamma$ . The statement (1) of the lemma follows.

(2) Suppose that  $\llbracket A \rrbracket \bigcirc \varphi \in \Delta$ . There are two cases to consider:

case 1:  $A \neq \mathbb{A}$ . We consider an arbitrary  $\sigma_A \in \text{act}_A(\Delta)$ . Then  $\sigma_A$  can be extended to a action vector  $\sigma' \sqsupseteq \sigma$ . Let  $N(\sigma_A)$  be the set  $\{i \mid \sigma_A(i) \geq m\}$ , where  $m$  is the number of enforceable successor formulae in  $\Delta$ , and let  $\text{co}(\sigma_A) = \left( \sum_{i \in N(\sigma_A)} (\sigma_A(i) - m) \right) \bmod l$ , where  $l$  is the number of unavoidable successor formulae in  $\Delta$ . Now, we consider  $\sigma' \sqsupseteq \sigma_A$  defined as follows:  $\sigma'_b = ((q - \text{co}(\sigma_A)) \bmod l + m$  and  $\sigma'_{a'} = m$  for any  $a' \in \mathbb{A} - (A \cup \{b\})$ , where  $b \in \mathbb{A} - A$ . Thus, we have  $\mathbb{A} - A \subseteq N(\sigma)$  and also  $\text{co}(\sigma') = (\text{co}(\sigma_A) + (q - \text{co}(\sigma_A))) \bmod l + m = q$ . Therefore, for this arbitrarily chosen  $\sigma_A$  there exists at least one state, say  $\Delta'$ , such that  $\Delta \xrightarrow{\sigma} \Delta'$  and  $\varphi \in \Delta'$ .

Case 2:  $A = \mathbb{A}$ . Then, by virtue of (H2),  $\langle\langle \emptyset \rangle\rangle \bigcirc \neg \varphi \in \Delta$  and thus, by the rule **(Next)**,  $\neg \varphi \in \Gamma$  for every successor  $\Gamma$  of  $\Delta$ . Then,  $\neg \varphi \in \Delta'$  for every  $\Delta'$  that is a successor of  $\Delta$  in  $\mathcal{T}^\eta$  and hence the colouring set of every leaf of  $\mathcal{T}$ . Then, the (unique) co- $\mathbb{A}$ -actions, which is an identity function, has the required properties.

The statement (2) of the lemma follows.  $\square$   $\square$

**Theorem 6.3.** *If  $\mathcal{T}^\eta$  is open, then  $\eta$  is satisfiable.*

*Proof.* The structure  $\mathfrak{F}$  constructed from  $\mathcal{T}^\eta$  is a Hintikka structure. Indeed, H1-H4 of Definition 6.1 are satisfied since the nodes of  $\mathfrak{F}$  are nodes of  $\mathcal{T}^\eta$ . H5 of the same definition essentially follows from Lemma 6.4. Whenever a node  $w$  of  $\mathfrak{F}$  contains a potential eventuality  $\xi$ , this means that this eventuality will stay in the queue (see construction of  $\mathfrak{F}$  above) until realized. Moreover, if the  $\mathcal{T}^\eta$ -tree  $\mathcal{W}$  chosen to complete  $\mathfrak{F}$  from  $w$  does not realize  $\xi$ , either  $\xi$  or one of its descendants is present in each newly generated dead-end of  $\mathfrak{F}$ . So, when it is the turn to realize  $\xi$  we add to each dead-end state the realizing  $\mathcal{T}^\eta$ -tree for  $\xi$ . This, together with Lemma 6.4, guarantees that there exists a realization witness tree for  $\xi$  on  $\mathfrak{F}$  at  $w$ . Thus, H6 of Definition 6.1 is satisfied, too.

By construction, the structure  $\mathfrak{F}$  is a concurrent game Hintikka structure for  $\eta$ , thus Theorem 6.1 can be applied to obtain from it a model for  $\eta$ . Thus  $\eta$  is satisfiable.  $\square$   $\square$

### 6.3 Complexity

**Theorem 6.4.** *The tableau procedure for  $\text{ATL}^+$  runs in 2EXPTIME.*

*Proof.* The argument generally follows the calculations computing the complexity of the tableau method for ATL in Section 4.7 of [9], with one essential difference:  $\|cl(\eta)\|$  for any ATL formula  $\eta$  is linear in its length  $|\eta|$ , whereas  $\|cl(\eta)\|$  for an  $ATL^+$  formula  $\eta$  can be exponentially large in  $|\eta|$ , as shown after Lemma 3.2. This exponential blow-up, combined with the worst-case exponential in  $\|cl(\eta)\|$  number of states in the tableau, accounts for the 2EXPTIME worst-case complexity of the tableau method for  $ATL^+$ , which is the expected optimal lower bound. It is also an upper bound for the tableau method, because no further exponential blow-ups occur in the prestate- and state-elimination phases.  $\square$   $\square$

There are various ways to restrict or parametrize the set of  $ATL^+$  formulae in order to avoid the exponential blow-up of their closure sets. As suggested by the example after Lemma 3.2, the main cause for that blow-up of the number of  $\gamma$ -components of  $\gamma$ -formulae  $\varphi = \langle\langle A \rangle\rangle\Phi$  or  $\varphi = \llbracket A \rrbracket\Phi$  in  $ATL^+$  is the nesting of conjunctions and disjunctions in the path formula  $\Phi$  which are not separated by temporal operators. Let us call the number of such nestings the *superficial Boolean depth* of  $\Phi$  and denote it by  $\delta_0(\Phi)$ . Then, let the *nested Boolean depth* of any  $ATL^+$  formula  $\Psi$ , denoted  $\delta(\Psi)$ , be the maximal superficial Boolean depth  $\delta_0(\Phi)$  of a path sub-formula  $\Phi$  of  $\Psi$ . For instance,  $\delta(\langle\langle 1 \rangle\rangle \bigcirc \langle\langle 1 \rangle\rangle (p \mathcal{U} \neg q)) = 0$ ,  $\delta(\langle\langle 1 \rangle\rangle (\Box p \vee ((q \wedge p) \mathcal{U} \neg q))) = 1$ ,  $\delta(\langle\langle 1 \rangle\rangle (\Diamond q \wedge (\Box p \wedge (q \mathcal{U} \neg q)))) = 2$ . Now, if this number for a formula  $\eta$  is bounded above, the size of the closure  $\|\eta\|$  becomes polynomially bounded in  $|\eta|$  because the nesting of  $\wedge$  and  $\vee$  when they are separated by a temporal operator does not have a multiplicative effect on the number of  $\gamma$ -components. Consequently, the complexity of the tableau method is reduced to single exponential time, caused only by the maximal possible number of states in the tableau, just like in ATL. Thus, we have the following.

**Proposition 6.1.** *The tableau procedure for  $ATL^+$  applied to a class of  $ATL^+$  formulae of bounded nested Boolean depth runs in EXPTIME.*

**Corollary 6.1.** *The tableau procedure for  $ATL^+$  applied to ATL formulae runs in EXPTIME.*

## 7 Concluding remarks

Here we have developed sound, complete and terminating tableau-based decision method for constructive satisfiability testing of  $ATL^+$  formulae and have argued for its practical usability and implementability. The method is amenable to further extension to the full  $ATL^*$ , but this is left to future work.

Some comparison with the automata-based method for satisfiability testing in  $ATL^*$ , presented in [12] are in order. The two methods appear to be quite different and, though eventually working in the same worst-case complexity, the double exponential blow-ups seem to occur in different ways, namely, in the automata-based method, one exponential blow-up occurs in converting the formula into an automaton, while the other is in the time complexity of checking non-emptiness of the resulting automaton. It would be instructive to

compare the practical implications and efficiency of both methods and we leave such systematic comparison to the future, when (hopefully) both methods are implemented. For now, we only mention that the formula  $\theta$  from our running example, the tableau for which is worked out explicitly and in detail in this paper, is translated with the method from [12] into an automaton with  $2^{12}$  alphabet symbols and over 100 states. Of course, this comparison cannot serve as an argument for general practical superiority in efficiency of the tableau-based method. Still, the technical details of both methods, illustrated in that example, indicate that, while the worst-case exponential blow-ups are bound to occur in both methods, they seem to be more controllable and avoidable in the tableau-based method, at the expense of its lesser automaticity and higher degree of user control. Thus, we would argue that both methods have generally incomparable pros and cons, and consequently are of independent interest, both theoretically and practically.

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